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On the Semiboundedness of the $(\phi^4)_2$ Hamiltonian*

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Abstract. An elementary alternate proof of the semiboundedness of the locally correct Hamiltonian $H_0 + \int :\phi^4(x): g(x)\,dx$ of the $(\phi^4)_2$ quantum field theory model. The interaction operator is expressed as the sum of a positive operator and operators which are "tiny" relative to N^ε for any $\varepsilon > 0$, where N is the number operator.

The semiboundedness of the space cut-off $(\phi^4)_2$ Hamiltonian was first proved by Nelson [1]. Alternative proofs and generalizations of this result have been given by various authors (see [2] and the references therein). In this note we give an elementary alternate proof in which the interaction operator $V = \int :\phi^4(x): g(x)\,dx (g \geqq 0,\ g \in L^1 \cap L^2)$ is expressed as the sum of a positive operator and operators which are "tiny" relative to N^ε for any $\varepsilon > 0$ (here N is the number operator). The proof is based on the formal identity $:\phi^4: = (:\phi^2: - 2c)^2 - 6c^2$ where c is the infinite constant $\int w(k)^{-1}\,dk$.

In our notation

$$a(k)\,a^+(p) - a^+(p)\,a(k) = \delta(k-p)$$

$$N = \int a^+(k)\,a(k)\,dk$$

$$H_0 = \int a^+(k)\,a(k)\,w(k)\,dk$$

$$\phi(x) = \int [a(k) + a^+(-k)] \exp(ikx)\,w(k)^{-1/2}\,dk$$

where $w(k) = (k^2 + m^2)^{1/2}$ and m is the mass of the free field ϕ.

Let $b > 0$ and define

$$f_n(k) = \begin{cases} w(k)^{-1/2} & |k| \leqq n^b \\ 0 & |k| > n^b . \end{cases}$$

Let $a_n = a_n(x) = \int a(k) \exp(ikx)\, f_n(k)\,dk$ and let a_n^+ be the adjoint of a_n and let

$$c_n = a_n a_n^+ - a_n^+ a_n = \|f_n\|^2 \quad \text{for} \quad n = 0, 1, \ldots .$$

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