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The Classical Limit of Quantum Spin Systems

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Abstract. We derive a classical integral representation for the partition function, Z^Q, of a quantum spin system. With it we can obtain upper and lower bounds to the quantum free energy (or ground state energy) in terms of two classical free energies (or ground state energies). These bounds permit us to prove that when the spin angular momentum $J \to \infty$ (but after the thermodynamic limit) the quantum free energy (or ground state energy) is equal to the classical value. In normal cases, our inequality is $Z^C(J) \leqq Z^Q(J) \leqq Z^C(J+1)$.

I. Introduction

It is generally believed in statistical mechanics that if one takes a quantum spin system of N spins, each having angular momentum J, normalizes the spin operators by dividing by J, and takes the limit $J \to \infty$, then one obtains the corresponding classical spin system wherein the spin variables are replaced by classical vectors and the trace is replaced by an integration over the unit sphere. Indeed, Millard and Leff [1] have shown this to be true for the Heisenberg model *when N is held fixed*. Their proof is quite complicated and it is therefore not surprising that this goal was not achieved before 1971. Despite that success, however, the problem is not finished. One wants to show that one can interchange the limit $N \to \infty$ with the limit $J \to \infty$, i.e. is the classical system obtained if we first let $N \to \infty$ and then let $J \to \infty$? In the Millard-Leff proof the control over the N dependence of the error is not good enough to achieve this desideratum.

A more useful result, and one which would include the above, would be to obtain, for each J, upper and lower bounds to the quantum free energy in terms of the free energies of two classical systems such that those two bounds have a common classical limit as $J \to \infty$. In this paper we do just that, and the result is surprisingly simple: In most cases of interest (including the Heisenberg model), the classical upper bound is

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