Commun. math. Phys. 31, 291—294 (1973)

A Continuity Property of the Entropy Density for Spin Lattice Systems

M. Fannes*
Universiteit Leuven, Belgium

Received December 15, 1972

Abstract. The entropy density of spin lattice systems is known to be a weak* upper semi-continuous functional on the set of the lattice invariant states. (It is even weak* discontinuous.) However we prove here that it is continuous with respect to the norm topology on those states.

I. Preliminaries

We consider a lattice $\mathbb{Z}^d$ of N spin states per lattice site. By $\Lambda \subset \mathbb{Z}^d$ we will always mean a non-empty finite volume and by $V(\Lambda)$ the number of points in it.

To $\Lambda \subset \mathbb{Z}^d$ we associate the local algebra $\mathscr{A}_\Lambda$ of observables:

$$\mathscr{A}_\Lambda = \mathscr{B}(\mathscr{H}_\Lambda) \quad \text{where} \quad \mathscr{H}_\Lambda = \bigotimes_{i \in \Lambda} \mathscr{H}_i \quad \text{and each} \quad \mathscr{H}_i$$

is an isomorphic copy of the N-dimensional Hilbert space $\mathbb{C}^N$. For $\Lambda_1 \subset \Lambda_2$ we trivially get an isometric embedding of $\mathscr{A}_{\Lambda_1}$ in $\mathscr{A}_{\Lambda_2}$ which maps A into $A \otimes 1_{\Lambda_2 \setminus \Lambda_1}$. This allows us to construct the C^*-algebra $\mathscr{A}$ of quasilocal observables:

$$\mathscr{A} = \overline{\bigcup_{\Lambda \subset \mathbb{Z}^d} \mathscr{A}_\Lambda}^{\,n} .$$

The natural translation mappings $\tau_x : \mathscr{A}_\Lambda \to \mathscr{A}_{\Lambda + x}$, $x \in \mathbb{Z}^d$ extend to a group $\{\tau_x \mid x \in \mathbb{Z}^d\}$ of automorphisms of $\mathscr{A}$. A state ω on $\mathscr{A}$ is called lattice invariant if $\omega \circ \tau_x = \omega$ for all $x \in \mathbb{Z}^d$. Let $\mathscr{E}$ denote the set of all lattice invariant states on $\mathscr{A}$.

For each state ω on $\mathscr{A}$ and for each $\Lambda \subset \mathbb{Z}^d$ there exists a unique density matrix $\varrho_\Lambda \in \mathscr{B}(\mathscr{H}_\Lambda)$ such that $\forall\, A \in \mathscr{A}_\Lambda$ $\omega(A) = \operatorname{Tr} \varrho_\Lambda A$.

The local entropy density of the state ω is given by

$$s_\Lambda(\omega) = -\frac{1}{V(\Lambda)} \operatorname{Tr} \varrho_\Lambda \log \varrho_\Lambda .$$

* Aspirant van het Belgisch N.F.W.O.