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Spectral and Scattering Theory for the Klein-Gordon Equation

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Abstract. Eigenfunction expansions associated with the Klein-Gordon equation, are derived in the static external field case. By employing these, we develop spectral and scattering theory. The results are almost as strong as those obtained in the Schrödinger case.

1. Introduction

We shall in this paper consider spectral and scattering theory for the Klein-Gordon (K-G) equation

$$(\square + 2iq_0(x)\,\partial_t - q_0(x)^2 + q_s(x) + m^2)\,u(x,t) = 0\,, \tag{1.1}$$

where $x \in R^3, t \in R$ and $\square = \partial_t^2 - \Delta$. The functions $q_0(x)$ and $q_s(x)$ are static external potentials coupled like a zero'th component of a four-vector and a scalar respectively.

Equations similar to (1.1) have been studied previously. Thoe [1] developed spectral and scattering theory in the case $m=0$, $q_0(x)=0$ and $q_s(x)>0$ by employing a method due to Lax and Phillips [2] (only suited for the $m=0$ case). Strauss [3] showed the existence and boundedness of the scattering operator and its inverse when $m>0$ and $q_0(x)=0$. Veselic [4] considered some spectral properties of Eq. (1.1) in the case $m>0$, $q_s(x)=0$ under very restrictive conditions on $q_0(x)$ (excluding for example the square well case).

We shall consider Eq. (1.1) when $m>0$ and derive eigenfunction expansions. These will enable us to develop the spectral theory of Eq. (1.1) in detail and develop scattering theory to the same level as is possible in the Schrödinger case [5–8].

The main motivation for this investigation (from a physical point of view) is the fact that once we have developed spectral and scattering theory for Eq. (1.1) when considered as a classical field equation, the associated quantum field theoretic problem can be completely solved (and will be considered elsewhere).

In Section 2 we start by specifying the class of potentials we shall consider. Furthermore, we write the K-G equation in the form $i\partial_t \Psi = A\Psi$