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The Ideal Boson Gas in an External Scalar Potential

E. B. Davies

Mathematical Institute, St. Giles, Oxford, U.K.

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Abstract. We consider a new way of going to the infinite volume thermodynamic limit for a finite density quantum system and apply it to the case of an ideal Boson gas. We describe two procedures for calculating the particle density in the thermodynamic limit, one local and one global, and show that they give different values for the density. Further calculations show that this discrepancy is caused by lack of macroscopic translation invariance of the system, which is not apparent at the microscopic level. We calculate the limiting value of the expectation function of the Weyl operators both above and below the critical density for Bose-Einstein condensation, and show that the condensate has paradoxical properties of a similar type to those recently discovered for the rotating Boson gas.

§ 1. Formulation of the Problem

We consider a system of spin-less particles, taking the single particle space to be $\mathscr{H} = L^2(\mathbb{R}^3)$. We suppose the particles are confined by an external scalar potential so that the single particle Hamiltonian is

$$H_L f(x) = -\tfrac{1}{2}\Delta f(x) + V(L^{-1}x) f(x) . \tag{1.1}$$

We take V to be non-negative with

$$\lim_{\|x\| \to \infty} V(x) = +\infty \tag{1.2}$$

so that the Hamiltonians are semi-bounded with discrete spectrum. In the limit $L \to \infty$ the Hamiltonian converges at least formally to a translation invariant free Hamiltonian. We denote the eigenvalues of H_L in increasing order by $\{E_{L,n}\}_{n=0}^{\infty}$ and the corresponding normalised eigenfunctions by $\{\phi_{L,n}\}_{n=0}^{\infty}$. We let $\mathscr{F}$ be the Boson Fock space over

$$\mathscr{F} = \sum_{n=0}^{\infty} \bigotimes_{\text{sym}}^{n} \mathscr{H} \tag{1.3}$$

and let $\sigma_{L,z}$ denote the trace class operator defined on $\mathscr{F}$ for $0 < z < 1$ and $\beta > 0$ by

$$\{\sigma_{L,z}\psi\}_n = \alpha_{L,z} z^n e^{n\beta E_{L,0}} \bigotimes^{n} e^{-\beta H_L} \psi_n \tag{1.4}$$