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On the Onsager-Yang-Value of the Spontaneous Magnetization

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Abstract. We show that the value of the spontaneous magnetization for the two-dimensional Ising model computed by Onsager is indeed, the appropriate derivative of the free energy with respect to the magnetic field. The argument is based on a simple application of the duality transformation.

1. Introduction

It is well known that Onsager never published his proof that the spontaneous magnetization in the Ising model is given by [1]

$$m_0(\beta) = \left(1 - \frac{1}{(\operatorname{sh} 2\beta J)^4}\right)^{1/8} \qquad \beta \geqq \beta_c \tag{1.1}$$

where β_c, determined by $\operatorname{sh} 2\beta_c J = 1$, is the critical inverse temperature ($\beta_c = 1/T_c$). J is the ferromagnetic coupling constant.

The above value was found by Yang [2] to coincide with

$$m_y(\beta) = \lim_{h\to 0^+} \lim_{N\to\infty} \lim_{M\to\infty} \frac{f_{NM}(\beta, h/N) - f_{NM}(\beta, 0)}{h/N} \tag{1.2}$$

and by Montroll, Potts, Ward [3] (see also footnote on p. 810 of [2]) to coincide with

$$m_A(\beta) = \lim_{|x-y|\to\infty} \lim_{N\to\infty} \lim_{M\to\infty} \sqrt{\langle \sigma_x \sigma_y \rangle_{P,NM}} \tag{1.3}$$

where f_{NM} and $\langle \sigma_x \sigma_y \rangle_{P,NM}$ denote, respectively, the free energy and the two spin correlation function of a rectangular lattice of $N \times M$ sites with periodic boundary conditions. h is the external magnetic field.

Although $m_y(\beta) \equiv m_0(\beta) \equiv m_A(\beta)$ it has never been proven [4] that these values coincide with the "true" spontaneous magnetization

$$m(\beta) = \left.\frac{\partial f(\beta, h)}{\partial \beta h}\right|_{h=0^+} \tag{1.4}$$

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