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Thermodynamic Duality for Classical Systems of Arbitrary Spin

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Abstract. Given a classical spin s system, namely, a set of spin sites of maximum spin s in ν-dimensional space along with a Hamiltonian defined on the possible spin configurations, a general method is described for constructing a large class of dual lattices of the same spin. The method utilizes the commutative group structure with which the configuration space is endowed.

In the classical statistical mechanics of spin $\frac{1}{2}$ lattice models, the set of spin configurations has a natural group structure, a fact that has been used by Sherman, for example, in a generalization of the Griffiths Inequalities [1]. In a similar manner, a group structure can be assigned to the set of spin configurations of a spin s lattice, for any s. Here a lattice refers loosely to any finite collection of spin sites in ν-dimensional space.

Duality, or the connection between high and low temperature properties for appropriately chosen pairs of lattices, has as applied to specific models a long history dating from the early work of Wannier and Onsager [2, 3]. Recently, Wegner has proved that any ferromagnetic spin $\frac{1}{2}$ lattice has a dual [4]. Merlini and Gruber have extended these results to arbitrary spin $\frac{1}{2}$ lattices by a constructive procedure [5].

In this article we generalize the construction of [5] to provide a family of duals for any lattice of arbitrary spin. In the first three sections, a group structure is introduced onto the space of configurations, and the dual groups are defined. For these systems the groups involved are simply products of the cyclic group of order $2s+1$. In fact the results can be extended to general finite abelian groups, corresponding to systems of mixed higher spins.

In the fourth section the dual interaction is derived for groups which are "non-π degenerate". This includes nearly all higher spin models of physical interest. Those groups not satisfying the restriction are dealt with in the following section.

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