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Conditions on a Connection to be a Metric Connection

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Abstract. It is shown that a torsion free linear connection is determined by a metric of given signature if and only if its holonomy group is a subgroup of the orthogonal group corresponding to the signature.

§ 1. Introduction

It is well known that a Riemannian metric g on a manifold M determines uniquely a torsion free linear connection Γ on M, called the Levi-Civita connection of g [1]. This connection is determined by the condition that parallel transport with respect to Γ should preserve the scalar product defined by g. The existence and uniqueness of Γ can be proved in various ways[1]. With respect to a local coordinate system (x^i) the Christoffel symbols of Γ are related to the components of the metric tensor by

$$g_{kl}\Gamma^l_{ji} = \tfrac{1}{2}(g_{ki|j} + g_{kj|i} - g_{ji|k}) \tag{1}$$

which is because of $\Gamma^i_{kl} = \Gamma^i_{lk}$ equivalent to

$$g_{hj|r} = g_{hi}\Gamma^i_{jr} + g_{jl}\Gamma^l_{hr}\,. \tag{2}$$

The purpose of this paper is to answer the following question: What are the necessary and sufficient conditions for a torsion free connection to be the Levi-Civita connection of a metric?

The most straight forward approach to this problem is to start with the differential equations (2) and write down the integrability conditions for the existence of a solution g_{ik} of (2)[2]. These integrability conditions form a system of functional equations $F^\nu(g_{ik}, \Gamma^i_{kl}, \Gamma^i_{kl|s_1\ldots s_j}) = 0$, $\nu = 1, 2 \ldots$ whose consistent solvability are necessary and sufficient for the existence of a solution of (2).

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[1] A very elementary proof: calculate $\frac{d}{dt}(g_{ik}(x^l(t))\,a^i(t)\,b^k(t)) = 0$ for $a^i(t)$, $b^k(t)$ parallel propagated along $x(t)$. Using $\frac{da^i}{dt} + \Gamma^i_{kl}\dot{x}^k a^l = 0$, $\frac{db^h}{dt} + \cdots$ one gets (2).

[2] This was done in some unpublished work by Müller zum Hagen.