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The Local b-Completeness of Space-times

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Abstract. It is shown that any point of space-time has a neighbourhood U such that the b-boundary $\dot{U}$ of U coincides with $\bar{U} \setminus U$.

1. Introduction

The b-boundary construction is a device to attach to any space-time a set of boundary points [1]. Such a boundary point can be considered as an equivalence class of inextensible curves in a space-time, whose affine length [2] (i.e. the length measured in a parallely propagated frame) is finite.

It may happen that such a curve is trapped in a compact set and still defines a "boundary point". An example is given by a closed null geodesic, which has moreover the following property. Choose a tangent vector X to the geodesic at a point p and parallelly propagate it along the geodesic. If we return to p with the vector λX, $0 < \lambda < 1$, then the affine length traversed in going round the geodesic n times is $l(1 + \lambda + \lambda^2 + \cdots + \lambda^{n-1})$. Hence the length of the inextensible curve defined by going round again and again is finite. Such a situation occurs in Taub-NUT space [2].

From the above example we learn that the following is possible: If U is an open submanifold of a space-time V^4 with compact closure $\bar{U}$ (relative to V^4), then $\dot{U}$, the b-boundary of the space-time U, possibly contains more points than $\bar{U} \setminus U$, the boundary of U relative to V^4. (From the definition of the b-boundary, $\bar{U} \setminus U \subset \dot{U}$ is obvious.)

Now one can ask the following question, which – as far as the author is aware – was first posed by Hawking: has any point $p \in V^4$ a neighbourhood U such that $\bar{U} \setminus U = \dot{U}$? An affirmative answer to this question is extremely important because otherwise the set of boundary points could be dense in the topological space $V^4 \cup \dot{V}^4$! Clearly the b-boundary would then be useless for a description of singularities.

The main purpose of this paper is to prove that $\bar{U} \setminus U = \dot{U}$ holds, provided there is no null geodesic trapped in $\bar{U}$.