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An Entropy Inequality for Quantum Measurements

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Abstract. It is proved that for an ideal quantum measurement the average entropy of the reduced states after the measurement is not greater than the entropy of the original state.

Consider a quantum state described by a density operator W in Hilbert space

$$W = W^* \geqq 0, \qquad \operatorname{Tr} W = 1 .$$

Then there exists an orthonormal set $\{|i\rangle\}$ such that

$$W = \Sigma w_i W_i$$

where $W_i = |i\rangle\langle i|$, $w_i > 0$, $\Sigma w_i = 1$.

Let O be an observable with eigenspaces defined by projections P_k;

$$O = \Sigma \omega_k P_k, \qquad \Sigma P_k = I .$$

If O is measured the value ω_k is obtained with probability $p_k = \operatorname{Tr} W P_k$ and W is then replaced by

$$W_k' = p_k^{-1} P_k W P_k .$$

The average over all possible outcomes gives a density operator

$$W' = \Sigma P_k W P_k = \Sigma p_k W_k'$$

(sum over all k such that $p_k \neq 0$).

The entropy of a state W is defined as

$$S(W) = -\operatorname{Tr} W \log W = -\Sigma w_i \log w_i .$$

It is well known that $S(W) \leqq S(W')$ [1] (the transformation $W \to W'$ is "dissipative") with equality if and only if $W = W'$. [A simple proof: use $\operatorname{Tr} W' \log W' = \operatorname{Tr} W \log W'$ and Klein's inequality $\operatorname{Tr}(W \log W - W \log W') \geqq 0$ ([2], p. 27).]

It was conjectured by Groenewold [3] that the average of the entropies of the states W_k' is not larger than $S(W)$:

$$S(W) - \Sigma p_k S(W_k') \geqq 0 .$$