

# Perturbations of the Modular Automorphism Group

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**Abstract.** It is shown, under a necessary condition, that strong (pointwise) convergence of modular automorphism groups to a one parameter family of maps implies weak convergence of the respective states in the factor case. Moreover the limiting one parameter family of maps is the modular automorphism group for the limiting state. In the type I case weak convergence of the automorphism groups suffices. Norm convergence of the states is obtained in some cases.

## 1.

One of the problems in the KMS theory which has so far been untouched is that of perturbation theory. This is the subject of what follows. We recall that if  $M$  is a von Neumann algebra with a one parameter automorphism group,  $\sigma_t$ ,  $-\infty < t < \infty$ , then a normal faithful linear functional  $\varphi$  on  $M$  is said to satisfy the KMS boundary condition (with respect to  $\sigma_t$ ) if for  $x, y \in M$  there exists a function  $F(z)$  holomorphic in, continuous and bounded on, the strip  $0 \leq \text{Im } z \leq 1$ , such that

$$F(t) = \varphi(\sigma_t(x) y) \quad \text{and} \quad F(t + i) = \varphi(y \sigma_t(x)). \quad (*)$$

This concept was first used in the algebraic approach by Haag, Hugenholtz and Winnink [6] and since then has been discussed by numerous authors, cf. [7, 9, 10, 12, 14–16, 18], and references therein. A result of Takesaki [14] is that *any* normal faithful state  $\varphi$ , on a von Neumann algebra  $M$  has a unique automorphism group satisfying (\*). This group is referred to as the modular automorphism group, an element of which is denoted  $\sigma_t^\varphi$ .

We want to determine the relation between the modular automorphism  $\sigma_t^\varphi$  and  $\varphi$ , viz., the extent, in some topological sense, to which the automorphism and the state depend on each other. This question is raised in [15, 16]. One manifestation of it, which we consider, is to show that when the automorphisms are “close”, so are the states. More precisely we show, under a necessary condition on states  $\psi_n$ , that strong (pointwise) convergence of the modular automorphism groups  $\sigma_t^{\psi_n}$  to a one parameter