Commun. math. Phys. 28, 87—108 (1972)

A Laplace Transform on the Lorentz Groups

I. Quasiregular Representations

N. W. Macfadyen

Department of Applied Mathematics and Theoretical Physics
University of Cambridge, England

Received April 20, 1972

Abstract. As a first step in the generalisation of the Laplace transform to a non abelian group, we examine the representations of the groups $SO(n, 1)$ by means of transformations of (not necessarily integrable) functions defined over the hyperboloids $O(n, 1)/O(n)$. We define a regularised version of the Gel'fand-Graev transformation from the n-dimensional hyperboloid to its associated cone, which is valid (under certain restrictions) for polynomially bounded functions. Upon the cone we then carry out a pair of classical Laplace transforms parallel to a generator. We give inversion formulas for both these procedures, and express the Laplace transform/inversion pair directly in terms of the function on the hyperboloid.

For integrable functions our results reduce to those already known; in the non-integrable case they are new. New features include the divergence of the transform for certain discrete asymptotic behaviours; the existence of a finite dimensional kernel subspace which is annihilated; good asymptotic behaviour of both Laplace projection and inversion formulas; and the existence of discrete terms contributing to the inversion formula for even dimension. Our results are valid for all dimensions and are completely independent of the usual "Laplace transforms" involving projection by means of "second-kind representation functions"; in a final section of the paper we examine briefly the significance of that approach in the light of our own.

I. Introduction

There has been recently [1, 2] considerable interest in possible generalisations of the Fourier transform on locally compact non-Abelian groups, with the hope of deriving expansion theorems valid for non-square-integrable functions over a non-compact group – typically one of the Lorentz groups $SO(n, 1)$. Two approaches to the problem can be distinguished: the distribution-valued-transform methods, as exemplified by (for instance) the work of Rühl [2]; and the special-function approach [1] which is often called the Laplace transform but which we shall call the Legendre transform. The former is the direct analogue of the classical one-dimensional theory; but it has the disadvantage that the function $f(g)$ with which we are concerned has to be regarded as a *distribution*, and although the Fourier transform is then certainly defined, there is no