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Inequalities for Traces on von Neumann Algebras

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Abstract. A number of useful inequalities, which are known for the trace on a separable Hilbert space, are extended to traces on von Neumann algebras. In particular, we prove the Golden rule, Hölder inequality, and some convexity statements.

A number of useful inequalities relating the traces of operators on a Hilbert space are known[1] when the trace is defined in the usual way. In this paper, we consider generalizations of some of these inequalities to traces on von Neumann algebras. In a subsequent paper, we will discuss applications to entropy and statistical mechanics.

In what follows τ will always be a normal, faithful[2] semifinite trace on a von Neumann algebra, $\mathfrak{A}$, of operators on a Hilbert space $\mathscr{H}$. This means that τ is a function, defined on $\mathfrak{A}^+ = \{A : A \geqq 0\}$ and extended to the 2-sided ideal, M, whose positive part is $M^+ = \{A : A \geqq 0 \text{ and } \tau(A) < \infty\}$ with the following properties[3]:

a) $\tau(A) \geqq 0$ if $A \geqq 0$. (1)

b) $\tau(A + \lambda B) = \tau(A) + \lambda\tau(B)$ if (2)
  i) λ in $\boldsymbol{C}$; A, B in M or,
  ii) $\lambda \geqq 0$; $A, B \geqq 0$.

c) $\tau(A) = \tau(UAU^*)$ if (3)
  $A \geqq 0$; U is unitary.

d) $\tau(AB) = \tau(BA)$ if (4)
  i) A in M, B in $\mathfrak{A}$ or,
  ii) $B = A^*$ in $\mathfrak{A}$.

e) (Normal): If $\{A_i\}$ is a bounded increasing net of positive operators, then $\sup_i \tau(A_i) = \tau\left(\sup_i A_i\right)$ (5)

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[1] See, for example [1–4].

[2] The restriction to faithful traces is not really necessary, (see [5], Corollary 2, p. 83) but simplifies things slightly.

[3] Properties (a), (b ii), and (c) suffice to define a trace (see [5], p. 81).