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Some Results in Non-Commutative Ergodic Theory

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Abstract. We study some properties of invariant states on a C^*-algebra $\mathscr{A}$ with a group G of automorphisms. Using the concept of G-factorial state, which is a "non-commutative" generalization of the concept of ergodic measure, in general wider in scope than G-ergodic state, we show that under a certain abelianity condition on $(\mathscr{A}, G)$, which in particular holds for the quasi-local algebras used in statistical mechanics, two different G-ergodic states are disjoint. We also define the concept of G-factorial linear functional, and show that under the same abelianity condition such a functional is proportional to a G-ergodic state. This generalizes an earlier result for complex ergodic measures.

1. Introduction

In a recent paper [1] we studied a possible extension of the concept of ergodic measure from the classical case of a positive measure to an arbitrary complex measure, requiring that for every G-invariant ($|m|$-a.e.) measurable subset E of the space X we have either $m(E)=0$ or $m(X-E)=0$. Here G is the group of transformations of X, and $|m|$ is the total variation of m. It turned out that this extension is essentially trivial, in the sense that such an ergodic measure m is of the form $k|m|$, with k a complex constant ("ergodicity implies positivity"). A related result – which, although it can be considered to be a direct corollary of the above result, is as easily proved directly from the extremality property of positive ergodic measures – is that two positive measures on the same space, ergodic under the same group, are either orthogonal (i.e. their supports are disjoint), or proportional. Namely, if m_1 and m_2 are two non-proportional positive ergodic measures, form $m=m_1+m_2$. Unless there is a measurable set E, G-invariant (m-a.e.), such that $m_1(E)=0$, $m_2(X-E)=0$, m is ergodic, which contradicts the non-trivial decomposition $m=m_1+m_2$. – Expressed in the C^*-algebra language, with $\mathscr{A}$ a C^*-algebra, acted on by a group G of automorphisms, this means that two different G-ergodic states on a commutative C^*-algebra are disjoint, i.e. the corresponding cyclic representations of $\mathscr{A}$ are disjoint. This