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On the Uniqueness of the Equilibrium State for Ising Spin Systems

JOEL L. LEBOWITZ*
Belfer Graduate School of Science, Yeshiva University, New York, N.Y.

ANDERS MARTIN-LÖF
Rockefeller University, New York, N.Y.

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Abstract. We show that for an Ising spin system of arbitrary spin with a ferromagnetic pair interaction and a "periodic" external magnetic field there is a unique equilibrium state if and only if the magnetization is continuous with respect to a uniform change in the external field. Hence, if the critical temperature T_c is defined as the temperature where the spontaneous magnetization (which is a non-increasing function of the temperature) becomes positive, then the equilibrium state is unique for $T > T_c$ and is non-unique for $T < T_c$ (when the external field is zero). This implies that the correlation functions have a cluster property for $T > T_c$.

We also show that for an anti-ferromagnet consisting of two sublattices there is a unique equilibrium state if and only if the staggered magnetization is continuous with respect to a change in the staggered field.

I. Introduction

We consider an Ising spin system with a ferromagnetic pair interaction in a finite box Λ on a d-dimensional lattice $\mathbb{Z}^d$, i.e. at each point p of the lattice there is a spin $\sigma_p = \pm 1$, and the conditional probability of a spin configuration in the box given a configuration outside it is proportional to

$$\exp(-E_\Lambda) = \exp\left(\tfrac{1}{2} \sum_{p \neq q \in \Lambda} J_{p-q}\sigma_p\sigma_q + \sum_{p \in \Lambda} \sigma_p \left(H_p + h + \sum_{q \notin \Lambda} J_{p-q}\sigma_q\right)\right). \quad (1)$$

$J_p = J_{-p} \geqq 0$ is the pair interaction, $\sum_{p \in \mathbb{Z}^d} J_p < \infty$, and $H_p + h$ an external magnetic field. The reciprocal temperature β has been included in the Hamiltonian. The external field consists of a uniform part h and a periodic part H_p, i.e. $H_p = H_{p+g}$ when g is contained in some subgroup G of $\mathbb{Z}^d$. A boundary condition for the box Λ is specified by giving a probability distribution $b_\Lambda(d\sigma)$ for the configurations outside Λ.

The (equilibrium) state of the system in Λ is the probability distribution for configurations in Λ defined by (1) together with b_Λ or equivalently

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