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Variational Principles and Spatially-Homogeneous Universes, Including Rotation*

M. A. H. MacCallum** and A. H. Taub

Department of Mathematics, University of California, Berkeley, Calif., USA

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Abstract. The validity of imposing spatial homogeneity on the variations in the usual action principle for Einstein's equations is studied. It is proved that with this procedure the standard and ADM Lagrangians give correct Einstein equations if and only if the space belongs to Class A of Ellis and MacCallum [1], i.e., the structure constants of the simply transitive group satisfy $C^f_{fg}=0$. The possibility of overcoming this difficulty in the Class B spaces is examined.

1. Introduction

When the source of the gravitational field described by the metric tensor ${}^4g_{\mu\nu}$ of space-time is a perfect fluid with stress-energy tensor[1]

$$T^{\mu\nu}=(w+p)\,u^\mu u^\nu+p\,{}^4g^{\mu\nu} \tag{1.1}$$

where p is the pressure, w the energy density and u^μ the fluid velocity ($u^\mu u_\mu=-1$) and when there is an equation of state of the form

$$w=w(p) \tag{1.2}$$

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** Permanent address: Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge, U. K. and King's College, Cambridge, U. K.

[1] We employ the following conventions. The signature of spacetime is +2 and units are such that $8\pi G=c=1$. The metric of space-time is denoted ${}^4g_{\mu\nu}$, the curvature and Einstein tensors by the usual $R_{\mu\nu}$, $G_{\mu\nu}$ etc., and covariant differentiation by a semi-colon separating indices. The corresponding quantities for an embedded three-space will be denoted by g_{ij}; $R^*{}_{ij}$, $G^*{}_{ij}$ etc.; and a bar separating indices. Greek indices run from 1 to 4, Latin indices from 1 to 3, x^4 being a time coordinate. A comma separating indices denotes partial derivative. Any undefined notation may be assumed to have its standard meaning. The sign conventions for the Riemann and Ricci tensors follow [1], i.e., for an arbitrary vector b^ν

$$b^\nu{}_{;\mu\tau}-b^\nu{}_{;\tau\mu}=-R^\nu{}_{\sigma\mu\tau}b^\sigma\,,$$

$$R_{\sigma\tau}=R^\nu{}_{\sigma\nu\tau};\ R=R^\sigma{}_\sigma\,.$$