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Hilbert Space Representations of Lie Algebras

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Abstract. We describe a new approach to the general theory of unitary representations of Lie groups which makes use of the Gelfand-Segal construction directly on the universal enveloping algebra of any Lie algebra. The crucial observation is that Nelson's theory of analytic vectors allows the characterisation of certain states on the universal enveloping algebra such that the corresponding representations of the universal enveloping algebra are the infinitesimal part of unitary representations of the associated simply connected Lie group. In the first section of the paper we show that with the aid of Choquet's theory of representing measures one can derive a simple new approach to integral decomposition theory along these lines.

In the second section of the paper we use these methods to study the irreducible unitary representations of general semi-simple Lie groups. We give a simple proof that the K-finite vectors studied by Harish-Chandra [5] are all analytic vectors. We also give new proofs of some of Godement's results [2] characterising spherical functions of height one, at least for unitary representations. Compared with [2] our method has the possible advantage of obtaining the characterisations by infinitesimal methods instead of using an indirect argument involving functions on the group. We point out that while being purely algebraic in nature, this approach makes almost no use of the deep and difficult theorems of Harish-Chandra concerning the universal enveloping algebra [5].

Our work is done in very much the same spirit as that of Power's recent paper [8]. The main difference is that by concentrating on a more special class of positive states we are able to carry the analysis very much further without difficulty.

§ 1. Representations of the Universal Enveloping Algebra

Let G be a real simply connected Lie group with Lie algebra $\mathscr{G}$ and let $\mathscr{U}(\mathscr{G})$ denoted the complex universal enveloping algebra of $\mathscr{G}$. An involution is defined on $\mathscr{U}(\mathscr{G})$ by supposing that $X^* = -X$ for all $X \in \mathscr{G}$.

If π is a unitary representation of G on a Hilbert space $\mathscr{H}$, we denote by $\mathscr{H}^\infty$ and $\mathscr{H}^\omega$ the spaces of C^∞- and analytic vectors respectively for π. It is known [6] that both of these are always dense subspaces of $\mathscr{H}$. There is a representation $\partial\pi$ of $\mathscr{G}$ by operators on $\mathscr{H}^\infty$ given by

$$\partial\pi(X)\xi = \lim_{t\to 0} t^{-1}\{\pi(\exp tX)\xi - \xi\}$$

and this has a natural extension to a representation $\partial\pi$ of $\mathscr{U}(\mathscr{G})$. The extension is a *-representation in the sense that for all $X \in \mathscr{U}(\mathscr{G})$

$$\partial\pi(X^*) \subseteq \partial\pi(X)^* .$$