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Correlation Inequalities on Some Partially Ordered Sets

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Abstract. We prove that increasing functions on a finite distributive lattice are positively correlated by positive measures satisfying a suitable convexity property. Applications to Ising ferromagnets in an arbitrary magnetic field and to the random cluster model are given.

1. Introduction

Recently, Griffiths obtained remarkable inequalities for the correlations of Ising ferromagnets with two-body interactions [1]. These inequalities were subsequently generalized to a larger class of spin systems [2, 5]. An apparently unrelated inequality for the probabilities of certain events in a percolation model had been derived earlier by Harris [6, Lemma (4.1)]. While Harris' inequality seems to have drawn less attention than it deserves, Griffiths' inequalities have received several applications of physical interest, and give useful information on the existence of the infinite volume limit and on the problem of phase transitions. Most interesting for the applications is the second inequality, which states that any two observables f and g in a suitably chosen class have positive correlations, or more precisely that their thermal averages, defined with a suitably restricted Hamiltonian, satisfy:

$$\langle fg\rangle - \langle f\rangle \langle g\rangle \geqq 0\,. \tag{1.1}$$

One of the simplest situations where a property of this type holds is the following. Let Γ be a finite totally ordered set, let μ be a positive measure on Γ. Define, for any function f on Γ

$$\langle f\rangle = Z^{-1} \sum_{x\in\Gamma} \mu(x)\, f(x) \tag{1.2}$$