Commun. math. Phys. 21, 211–218 (1971)

New Elliptic System and Global Solutions for the Constraints Equations in General Relativity

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Received November 1, 1970

Abstract. By a new choice of the arbitrarily given quantities on an initial 3-manifold we reduce the system of constraints, in General Relativity, to an elliptic system of four equations, the coefficients of which have a simple geometric interpretation on the 3-manifold. The system seems well suited for a global study and some results are given in this direction.

Introduction

An initial data set $\mathscr{I}$, in General Relativity[1], is a 3 dimensional differentiable manifold V_3, a negative definite riemannian metric on $V_3, \bar{\boldsymbol{g}}$, and a second rank, symmetric, tensor $\boldsymbol{P}$, satisfying the constraint equations:

$$\bar{\nabla}_j(P_i^j - \delta_i^j P) = 0, \qquad P = P_i^i,$$
$$\bar{R} + H^2 - P^2 = 0, \qquad H^2 = P_{ij}P^{ij}.$$

Where $\bar{R}$ is the riemannian scalar curvature, and $\bar{\nabla}$ the covariant derivative of $\bar{\boldsymbol{g}}$.

An einsteinian space time is a 4-dimensional differentiable manifold V_4, endowed with a hyperbolic metric $\boldsymbol{g}$, with vanishing Ricci tensor. It is said that $(V_4, \boldsymbol{g})$ is a solution of the Cauchy problem, associated with the initial data $\mathscr{I}$, if there is a diffeomorphism Λ of V_3 with a submanifold Σ of V_4 such that the image by Λ of $\bar{\boldsymbol{g}}$ and $\boldsymbol{P}$ coïncide respectively with the metric induced on Σ by Λ and the second fundamental form of Σ as submanifold of $(V_4, \boldsymbol{g})$.

It is known that to each initial data set corresponds an einsteinian space time, and only one in the class of maximal, globally hyperbolic space times [4].

The system of constraint equations (1), (2) has been already extensively studied[2], but rather few global (i.e. interesting) solutions are

[1] We speak, for simplicity, of empty-space equations. We will, in an appendix, show how the interior case can be treated along the same lines.

[2] For a bibliography up to 1962 see Bruhat (1962).