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Hamiltonians Defined as Quadratic Forms*

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Abstract. We present a complete mathematical theory of two-body quantum mechanics for a class of potentials which is larger than the usual L^2-classes and which includes potentials with singularities as bad as $r^{-2+\varepsilon}$. The basic idea is to define $H_0 + V$ as a sum of quadratic forms rather than as an operator sum.

§ 0. Introduction

After one has established the connection between quantum mechanics and Hilbert space objects [1], a host of nontrivial mathematical questions arise. These involve establishing the self-adjointness of the Hamiltonian, studying qualitative properties of bound states, investigating the question of the existence of the limits needed for scattering theory, and proving the physically expected properties of the S-matrix. For the two-body case, $H = -\Delta + V$, with V in one of the four classes, $L^2 + L^\infty$, $L^2 + (L^\infty)_\varepsilon$[1], L^2, $L^2 \cap L^1$ there is a fairly complete theory of these questions [2] going back to the famous paper of Kato [3]. Our purpose here is to extend this theory to a larger class of potentials than the L^2 classes. While one can establish some n-body results for these larger classes (e.g. Hunziker's theorem on the position of the continuum [4] goes through), we only discuss the two-body case here – a case for which a complete theory exists. A fuller discussion of this theory can be found in [5]; in this note we wish to emphasize the main results and the physics behind these results.

Before describing the classes we treat in detail, let us explain why such larger classes should exist. Consider a potential $V = r^{-\alpha}$. This is in an L^2 class only if $0 \leqq \alpha < 3/2$ but physically there is nothing singular about $-\Delta - r^{-3/2}$. It is only at $\alpha = 2$ that singular things begin to happen. At $\alpha = 2$, the uncertainty principle "proof" that H is bounded below breaks down (and in fact $-\Delta - cr^{-2}$ is not bounded below if $c > 1/4$).

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[1] $X + (L^\infty)_\varepsilon = \{f \mid (\forall \varepsilon) f = x_\varepsilon + g_\varepsilon \text{ with } x_\varepsilon \in X;\ \|g_\varepsilon\|_\infty < \varepsilon\}$; e.g. $r^{-1} \in L^2 + (L^\infty)_\varepsilon$ on $\mathbb{R}^3$.