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Hamiltonian Structures for Homogeneous Spaces

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Abstract. The definition and classification of classical relativistic particles requires the classification of certain invariant tensor fields on the inhomogeneous Lorentz group. The entire 10-parameter set is exhibited. At the same time, a much larger class of Lie groups is treated. The connection with particles will be presented in the succeeding article.

1. Introduction

The action of *time* in the phase space of a dynamical system can for Hamiltonian systems be carried out by "canonical" transformations.

In an Einstein-Lorentz invariant system, the entire space-time group (Poincaré group $\mathscr{P}$) acts in the phase space. Those systems where one has *or can introduce* a suitable Poisson bracket operation preserved by the entire group and not merely the time are of special interest. Here the cases in which the group acts *transitively* are of basic interest. This problem is what we study here.

The phase space in these cases has the form G/Γ where Γ is a closed subgroup of the "space-time" group G. Let $C^2(G;\Gamma)$ be the closed invariant 2-forms on G for which the Lie algebra of Γ is singular, but which are non-singular on the residue linear space of the Lie algebra of G *modulo* that of Γ. Then $C^2(G;\Gamma)$ corresponds $1:1$ to the possible Poisson brackets.

It is shown that for many pairs G, Γ the "generating functions" (in particular the Hamiltonian, which "generates" time translation) are linear combinations of matrix elements from the adjoint representation of G. This includes the case of the Poincaré group.

The application of these findings to the classification of elementary particles will be published separately. It requires taking into account which pairs of Poisson brackets are equivalent.

2. Alternating Structures

Let M be a differential manifold [4]. Suppose A is a contravariant tensor field of order 2. In terms of coordinates $x^1, \ldots, x^n$ for M one can define $\{f, g\} = A^{ij} f_i g_j$ for any two functions f, g defined on M where A^{ij} are the components of A, f_i is $\partial f/\partial x^i$ and g_j is $\partial g/\partial x^j$. Suppose A has the