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Self-Adjoint Algebras of Unbounded Operators*

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Abstract. Unbounded *-representations of *-algebras are studied. Representations called self-adjoint representations are defined in analogy to the definition of a self-adjoint operator. It is shown that for self-adjoint representations certain pathologies associated with commutant and reducing subspaces are avoided. A class of well behaved self-adjoint representations, called standard representations, are defined for commutative *-algebras. It is shown that a strongly cyclic self-adjoint representation of a commutative *-algebra is standard if and only if the representation is strongly positive, i.e., the representations preserves a certain order relation. Similar results are obtained for *-representations of the canonical commutation relations for a finite number of degrees of freedom.

Introduction

In this paper we study unbounded *-representations of *-algebras. The basic definitions, notation and motivation are drawn from the Wightman formulation of quantum field theory and the theory of Lie algebras. The general plan of the paper is to examine some of the pathologies associated with *-algebras of unbounded operators and, then, to find natural definitions which rule out these pathologies. Two such definitions are those of self-adjointness for representations (Section IV) and strong positivity (Sections VII and VIII).

The results of each section are summarized at the beginning of each section. We claim little or no originality for the contents of Sections I, II, V and VI which consist largely of background material, definitions modified from C^*-algebra theory and quantum field theory and known examples illustrating features of unbounded representations.

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