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Correlations between Eigenvalues of a Random Matrix

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Abstract. Exact analytical expressions are found for the joint probability distribution functions of n eigenvalues belonging to a random Hermitian matrix of order N, where n is any integer and $N \to \infty$. The distribution functions, like those obtained earlier for $n = 2$, involve only trigonometrical functions of the eigenvalue differences.

I. Statement of Results

A finite stretch of eigenvalues $E_1, E_2, \ldots, E_r$ of a random Hermitian matrix H of order $N \gg r$ has a well-defined statistical behavior in the limit as $N \to \infty$. A convenient way to discuss this behavior is to relate the eigenvalues E_j to the angles θ_j belonging to a certain *Circular Ensemble* [1, 2]. If D is the mean level-spacing of the eigenvalue series, we write

$$\theta_j = \frac{2\pi}{ND} E_j, \quad j = 1, \ldots, r, \tag{1.1}$$

and take for the complete series of angles $(\theta_1, \ldots, \theta_N)$ the probability distribution

$$Q_{N\beta}(\theta_1, \ldots, \theta_N) = C_{N\beta} \prod_{j<k} |e^{i\theta_j} - e^{i\theta_k}|^\beta, \tag{1.2}$$

where $\beta = 1, 2$ or 4. The case $\beta = 1$ applies to the usual physical situation in which H is real and symmetric, in particular when H is invariant under time-reflection and under space-rotations. The case $\beta = 2$ would apply when H is complex Hermitian, i.e. when there is no time-reflection invariance. The case $\beta = 4$ would apply when H is invariant under time-reflection, without any rotation-invariance, for a system with half-integer spin. Until now no interesting physical examples have been found of the cases $\beta = 2$ and 4. The case $\beta = 1$ has been extensively studied in connection with the statistics of neutron capture levels in heavy nuclei [3–6].