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Entropy Inequalities

HUZIHIRO ARAKI
Research Institute for Mathematical Sciences, Kyoto University, Kyoto, Japan

ELLIOTT H. LIEB*
Department of Mathematics
Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA

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Abstract. Some inequalities and relations among entropies of reduced quantum mechanical density matrices are discussed and proved. While these are not as strong as those available for classical systems they are nonetheless powerful enough to establish the existence of the limiting mean entropy for translationally invariant states of quantum continuous systems.

I. Introduction

In this note we shall be concerned with inequalities satisfied by the entropies of reduced density matrices. We begin with some definitions and a statement of our main Theorem 1. Section II contains the proof of the main theorem when the dimension is finite. Section III contains some other inequalities that can be derived from Theorem 1 by application of certain transformations. Section IV contains the proof of the main theorem when the dimension is infinite. Section V deals with the application of our theorem to the existence of the mean entropy for translationally invariant states of a quantum continuous system.

Definition 1. *A density matrix, ϱ, on a Hilbert space, H, is a self adjoint non-negative trace class operator on H whose trace is unity.*

Definition 2. *If ϱ is a density matrix,*

$$S(\varrho) = -\operatorname{Tr} \varrho \ln \varrho \tag{1.1}$$

is the entropy associated with ϱ.

Since $0 \leqq \varrho \leqq 1$, we have $-e^{-1} \leqq \varrho \ln \varrho \leqq 0$ and $(\psi_j, (\varrho \ln \varrho)\psi_j) \leqq 0$ for any ψ_j. Hence

$$S = -\sum_j (\psi_j, \varrho \ln \varrho \, \psi_j) \tag{1.2}$$

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