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Geometrodynamics of Electromagnetic Fields in the Newman-Penrose Formalism*

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Abstract. The "already unified" field theory of Rainich, Misner, and Wheeler is rederived in the spin-coefficient formalism of Newman and Penrose. Conditions equivalent to the Rainich algebraic conditions are obtained by classifying the tracefree Ricci tensor according to its principal null directions. The case of a null electromagnetic field is also treated fully. Necessary and sufficient conditions are given for a Riemannian geometry to have an electromagnetic field, null or non-null, as its source.

1. Introduction

Many attempts have been made to formulate a unified field theory, that is, one which treats both gravitation and electromagnetism as aspects of the geometry of space-time. These attempts usually take the direction of changing the geometry used in Einstein's theory to accommodate the electromagnetic field as well as the gravitational one. However, Rainich, Misner and Wheeler [1–3] have shown that Einstein's theory is "already unified". Given a Riemannian geometry (with metric tensor $g_{\mu\nu}$, Ricci tensor $R_{\mu\nu}$ and Ricci scalar R) they have found the following necessary and sufficient conditions for this geometry to have a non-null electromagnetic field as its source:

$$R = 0\,, \tag{1.1}$$

$$R_{\mu\alpha}R^{\alpha}{}_{\nu} = 1/4(R_{\tau\beta}R^{\tau\beta})g_{\mu\nu}\,, \tag{1.2}$$

$$R_{\mu\nu}W^{\mu}W^{\nu} \leqq 0 \quad \text{for any time-like vector } W^{\mu}\,, \tag{1.3}$$

$$R_{\tau\beta}R^{\tau\beta} \neq 0\,, \tag{1.4}$$

$$V_{\lambda;\tau} - V_{\tau;\lambda} = 0\,, \tag{1.5}$$

where

$$V_{\lambda} = \frac{\sqrt{-g}\,\varepsilon_{\lambda\nu\beta\gamma}R^{\beta\mu;\nu}R_{\mu}{}^{\gamma}}{R_{\varrho\kappa}R^{\varrho\kappa}}\,. \tag{1.6}$$

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