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Metrics on Test Function Spaces for Canonical Field Operators

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Abstract. In a canonical field theory, the field $\Phi(f)$ and momentum $\pi(g)$ are assumed defined for test functions f and g which are elements of linear vector spaces $\mathscr{V}_\Phi$ and $\mathscr{V}_\pi$, respectively. Generally, the continuity of the map onto the unitary Weyl operators $U(f)$, $V(g)$ is taken as ray continuity, the barest minimum to recover the field operators as their generators, i.e., $U(f) = e^{i\Phi(f)}$, $V(g) = e^{i\pi(g)}$. This leaves open the question of whether any wider continuity properties follow and what form they would take. We show that much richer continuity properties do follow in a natural fashion for every cyclic representation of the canonical commutation relations. In particular, we show that the test function space may be taken as a metric space, that the space may be uniquely completed in this topology, and that the map into the unitary Weyl operators is strongly continuous in this topology. The topology induced by this metric is minimal in the sense that it is the weakest vector topology for which the maps $f \to U(f)$, $g \to V(g)$ are strongly continuous. An expression for a suitable metric can easily be given in terms of a simple integral over a state on the Weyl operators.

Introduction

By smearing the canonical field operators $\Phi(x)$ and $\pi(x)$ in the usual way with test functions from spaces $\mathscr{V}_\Phi$ and $\mathscr{V}_\pi$ and then going over to the unitary Weyl operators one arrives at the customary definition of a representation of the canonical commutation relations (CCR) (cf., e.g., [9] or [10]). The only continuity requirement is ray continuity, i.e., in the notation of [1], $U(\lambda f)$ and $V(\lambda g)$ are assumed to be weakly continuous in λ. One has $V(g)\, U(f) = \exp\{i(f, g)\}\, U(f)\, V(g)$ where (f, g) is the nondegenerate inner product between $\mathscr{V}_\Phi$ and $\mathscr{V}_\pi$.

The generality and abstractness of this formulation leaves quite open questions about the role of the spaces $\mathscr{V}_\Phi$ and $\mathscr{V}_\pi$, the relevance of topologies they may carry, the possibility of a wider continuity of the representation than just ray continuity, whether the spaces $\mathscr{V}_\Phi$ and $\mathscr{V}_\pi$ can be enlarged, etc.

The purpose of this paper is to clarify some of these properties of test function spaces for CCR representations, and specifically to show that there exists a simple and natural metric $d(f, g)$ determined by the representation of the Weyl operators and consistent with the linear vector