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Stability of General Relativistic Gaseous Masses and Variational Principles*

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Abstract. The Einstein field equations for a self-gravitating fluid that obeys an equation of state of the form $p = p(w)$, p the pressure and w the energy density may be derived from a variational principle. The perturbations of the metric tensor and the fluid dynamic variables satisfy equations which may be derived from a related variational principle, namely the principle associated with the "second variation problem." It is shown that the variational principle given by Chandrasekhar from which a sufficient criterion may be obtained for deciding when a self gravitating spherical gaseous mass is unstable against spherically symmetric perturbations is that given by the "second variation problem". It is further shown that this criterion is equivalent to requiring that the integral entering into the second variation be negative. The latter form of the criterion may be used in general situations.

1. Introduction

It is the purpose of this paper to apply the variational principle [1] obeyed by self-gravitating fluids which satisfy an equation of state to the discussion of the stability against radial perturbations of a spherically symmetric distribution of such a fluid. We shall show that the variational principle given by Chandrasekhar [2] for determining the stability of a spherically symmetric self gravitating gaseous mass is given by the "second variation problem" associated with the principle referred to above.

Such a result is to be expected for it is well knwon that the equations satisfied by perturbations of solutions of the Euler equations of a variational principle are the Euler equations of another variational principle – the second variation problem. The two variational problems are related as follows: Let $\mathscr{L}(\phi^A; \phi^A_{,\mu})$ be a scalar density formed from some scalar or tensor fields ϕ^A and the derivatives of these fields with respect to the coordinates in space time,

$$\phi^A_{,\mu} = \frac{\partial \phi^A}{\partial x^\mu} \qquad \begin{array}{l} \mu = 1, 2, 3, 4 \\ A = 1, 2, \ldots, N . \end{array}$$

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