

## A Remark on $C^*$ -Algebras

HORST BEHNCKE\*

Department of Physics, Indiana University, Bloomington, Indiana  
Courant Institute of Mathematical Sciences, New York University, New York

Received October 10, 1968

**Abstract.** It is shown that any complex Banach algebra with hermitean involution and the weak  $C^*$ -property  $|x|^2 = |x^2|$  for all  $x = x^*$  is a  $C^*$ -algebra.

Among all complex Banach algebras with involution the  $C^*$ -algebras are singled out by a very strong condition on the norm. ONO [1], VIDAV [2] and more recently BERKSON [3] were able to replace the original Gelfand-Naimark axioms by considerably weaker assumptions. In this note we present a new proof for generalizations of the results in [2] and [3]. In particular we shall study complex Banach algebras  $\mathfrak{A}$  with involution  $*$  and norm  $|\cdot|$ , which also satisfy:

- a) the involution is hermitean and
- b)  $|x^2| = |x|^2$  for all  $x \in \mathfrak{A}_h$ .

Here  $\mathfrak{A}_h$  denotes the system of hermitean elements of  $\mathfrak{A}$ . We shall show that in this case  $\mathfrak{A}$  is a  $C^*$ -algebra. In contradistinction to [2] and [3] we shall not assume the existence of an identity. We also do not assume the involution to be isometric. This result is of importance in algebraic physics, because we only assume conditions on the hermitean elements, which correspond to observables. Mathematically our result shows that the  $C^*$ -property is a local property.

The terminology of [4] will be used throughout this paper. We begin with the simpler case of a commutative Banach algebra.

**Theorem 1.** *Any complex commutative Banach algebra  $\mathfrak{A}$  with involution, which satisfies a) and b) is a  $C^*$ -algebra.*

*Proof.* i) Because of a)  $\mathfrak{A}$  is symmetric. Thus its  $*$ -radical  $\mathfrak{R}$  consists of all  $x$  with  $\text{Sp}xx^* = \{0\}$ . This implies by b)  $\mathfrak{R} = \{x | xx^* = x^*x = 0\} = \{0\}$ . Thus  $\mathfrak{A}$  is  $*$ -semi simple, and the Gelfand representation defines an isomorphism of  $\mathfrak{A}$  with an algebra of functions  $\mathfrak{B}$ . The supremum norm of  $\mathfrak{B}$  can be carried back to  $\mathfrak{A}$  and defines an auxiliary norm  $|\cdot|_0$ , which makes  $\mathfrak{A}$  an  $A^*$ -algebra. Because of b) we have  $|x|_0 = |x|$  for all  $x \in \mathfrak{A}_h$ . Thus  $|x|_0 \leq |x| \leq \frac{1}{2}|x + x^*| + \frac{1}{2}|x - x^*| = \frac{1}{2}|x + x^*|_0 + \frac{1}{2}|x - x^*|_0 \leq 2|x|_0$  and we see that  $|\cdot|_0$  and  $|\cdot|$  are equivalent. In particular  $\mathfrak{A}$  equipped with  $|\cdot|_0$  is a  $C^*$ -algebra.

\* The research in this paper was partially supported by the U. S. Army Research Office, Durham.