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A Class of Homogeneous Cosmological Models

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Abstract. Einstein's field equations are studied under the assumptions that (1) the source of the gravitational field is a perfect fluid, and (2) there exists a group of motions simply transitive on three-surfaces orthogonal to the fluid flow vector. There are two classes of solutions; these are studied in detail. Three special families of solutions examined include all analytic solutions of the field equations obeying (1) and (2) of which the authors are aware. The relation of these solutions to various vacuum solutions is indicated.

1. Introduction

We shall consider solutions of Einstein's field equations

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = T_{ab} \tag{1.1}$$

in which the matter tensor takes the form of a perfect fluid

$$T_{ab} = \mu u_a u_b + p(g_{ab} + u_a u_b)\,, \qquad u_a u^a = -1 \tag{1.2}$$

where u^a is the normalised four-velocity, μ the density and p the pressure of the fluid. We shall normally assume $\mu + p > 0$. The Eqs. (1.1) and (1.2) are integrable provided we are given an equation of state. This we will usually assume to have the form $p = p(\mu)$. (Because of the homogeneity, this will be no restriction on our models unless μ takes the same value twice.)

Exact analytic solutions of these equations have, of necessity, high symmetry. The conservation equation $u_a T^{ab}_{\ ;b} = 0$ takes the form

$$\mu^{\cdot} := \mu_{,a} u^a = -(\mu + p)\,\theta \tag{1.3}$$

where $\theta := u^a_{\ ;a}$ is the expansion of the fluid (see Refs. [1, 3] for standard notation and results). Thus homogeneity of space-time, which implies a constant density μ, also implies that $\theta = 0$ and so we would not see an almost isotropic redshift. Hence such spacetimes are not reasonable cosmological models. The two simplest classes of spacetime that will give reasonable cosmological models are the well-known Friedmann universes, which are isotropic and are homogeneous on spacelike sections, and those