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On Weak and Monotone σ-Closures of C^*-Algebras

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Abstract. It is proved that the monotone σ-closure of the self-adjoint part of any C^*-algebra A is the self-adjoint part of a C^*-algebra $\mathscr{B}$. If A is of type I it is proved that $\mathscr{B}$ is weakly σ-closed, i.e. $\mathscr{B}$ is a Σ^*-algebra. The physical importance of Σ^*-algebras was explained in [1] and [7].

We recall that the class of bounded real Baire functions $\mathscr{B}^R(X)$ on a locally compact Hausdorff space X is defined as the monotone σ-closure of $C_0^R(X)$. It is immediately verified that $\mathscr{B}^R(X)$ is closed under pointwise limits of sequences hence $\mathscr{B}^R(X)$ is also the weak (pointwise) σ-closure of $C_0^R(X)$.

Regarding a C^*-algebra A as the non-commutative analogue of some $C_0(X)$ we may for a convenient representation of A as operators on a Hilbert space H form the monotone σ-closure $\mathscr{B}_A^R$ of A^R in $B(H)$. This class of Baire operators was introduced in [5] by R. V. KADISON in order to give measure-theoretic conditions on a representation between two concrete C^*-algebras to have a normal extension. His result together with those of [6] seem to indicate that $\mathscr{B}_A^R$ is able to take over the rôle played by the Baire functions in commutative theory.

Recently E. B. DAVIES in [1], [2] and [3] has considered instead the weak σ-closure of A and has outlined an interesting theory of Σ^*-algebras i.e. C^*-algebras which are weakly σ-closed. Since for non-commutative C^*-algebras one cannot use lattice arguments it is no more an easy matter to determine whether the weak and monotone σ-closure of A^R coincide. We prove in this paper that such is indeed the case if A is of type I. Unfortunately the proof will not be applicable for other types but since we are able to show in general that $\mathscr{B}_A^R$ is the self-adjoint part of a C^*-algebra we feel rather optimistic that the result is true in general i.e. that $\mathscr{B}_A^R + i\mathscr{B}_A^R$ is a Σ^*-algebra.

We shall use [4] as a standard reference on notations and terminology. In particular for a C^*-algebra A we shall write A'' for the enveloping von Neumann algebra of A in its universal representation. When no confusion may arise we shall drop the subscript and write $\mathscr{B}^R$ for the monotone σ-closure of A^R in A''.

Theorem 1. *$\mathscr{B}^R$ is the self-adjoint part of a C^*-algebra.*