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Semimodularity and the Logic of Quantum Mechanics*

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Abstract. If $(\mathscr{E}, \mathscr{S}, P, \Omega)$ is an event-state-operation structure, then the events form an orthomodular ortholattice $(\mathscr{E}, \leqq, ')$ and the operations, mappings from the set of states $\mathscr{S}$ into $\mathscr{S}$, form a Baer *-semigroup $(S_\Omega, \circ, *, ')$. Additional axioms are adopted which yield the existence of a homomorphism θ from $(S_\Omega, \circ, *, ')$ into the Baer *-semigroup $(S(\mathscr{E}), \circ, *, ')$ of residuated mappings of $(\mathscr{E}, \leqq, ')$ such that $x \in S_\Omega$ maps states while $\theta_x \in S(\mathscr{E})$ maps supports of states. If $(\mathscr{E}, \leqq, ')$ is atomic and there exists a correspondence between atoms and pure states, then the existence of θ provides the result: $(\mathscr{E}, \leqq, ')$ is semimodular if and only if every operation $x \in S_\Omega$ is a pure operation (maps pure states into pure states).

0. Introduction

The theory of orthomodular ortholattices provides the mathematical constructs for the quantum logic approach to the foundations of quantum physics. A role for the theory of Baer *-semigroup, a mathematical theory closely related to the theory of orthomodular ortholattices, was exhibited in [15]. The definitions and terminology introduced in [15] will be utilized in this paper without further explanation. If $(\mathscr{E}, \mathscr{S}, P, \Omega)$ is an event-state-operation structure, then $(\mathscr{E}, \leqq, ')$ is an orthomodular ortholattice and $(S_\Omega, \circ, *, ')$ is a Baer *-semigroup such that $p \in \mathscr{E} \to \Omega_p \in P'(S_\Omega)$ is an isomorphism of $(\mathscr{E}, \leqq, ')$ onto the orthomodular ortholattice $(P'(S_\Omega), \leqq, ')$ of closed projections in S_Ω. Each $x \in S_\Omega$ is a mapping, $x : \mathscr{D}_x \to \mathscr{R}_x$, with domain $\mathscr{D}_x$ and range $\mathscr{R}_x$ contained in $\mathscr{S}$.

The connection between the theories of orthomodular ortholattices and Baer *-semigroups includes the following: if $(L, \leqq, ')$ is any orthomodular ortholattice, then there exists a Baer *-semigroup $(S(L), \circ, *, ')$ where $S(L)$ is a set of mappings of L into L and there exists an injective mapping $j : L \to S(L)$. Section I is devoted to a discussion of $S(\mathscr{E})$ for the orthomodular ortholattice $(\mathscr{E}, \leqq, ')$. In particular, the relation of $(S(\mathscr{E}), \circ, *, ')$ to the Baer *-semigroup $(S_\Omega, \circ, *, ')$ of operations will be exhibited.

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