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Geometry of Quantum States

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Abstract. In the first part of this work, an attempt of a realistic interpretation of *quantum logic* is presented. Propositions of quantum logic are interpreted as corresponding to certain macroscopic objects called filters; these objects are used to select beams of particles. The problem of representing the propositions as projectors in a Hilbert space is considered and the classical approach to this question due to Birkhoff and von Neumann is criticized as neglecting certain physically important properties of filters. A new approach to this problem is proposed.

The second part of the paper contains a revision of the concept of a state in quantum mechanics. The set of all states of a physical system is considered as an abstract space with a geometry determined by the transition probabilities. The existence of a representation of states by vectors in a Hilbert space is shown to impose strong limitations on the geometric structure of the space of states. Spaces for which this representation does not exist are called non-Hilbertian. Simple examples of non-Hilbertian spaces are given and their possible physical meaning is discussed. The difference between Hilbertian and non-Hilbertian spaces is characterized in terms of measurable quantities.

1. Introduction

One of the fundamental assumptions of quantum mechanics is that quantum states may be represented by vectors in a linear space. For Dirac this assumption was a sort of a guess suggested by the nature of the superposition principle. It leads to representing pure states either by vectors in a Hilbert space or by distributions (i.e., unnormalizable vectors like e.g. plane waves.) The collection of these concepts forms a language in which quantum mechanics describes microphenomena.

It sometimes happens that the physical reality may not be expressed in terms of certain concepts if some applicability conditions do not hold. Thus, e.g., a field of forces cannot be described in terms of a potential if the curl does not vanish. In the Riemannian space the Cartesian coordinates may not be introduced if the space is not flat. The question arises whether the representation of quantum states by vectors does not impose certain limitations on the admissible structure of states.

A well known approach to this problem has been originated by Birkhoff and von Neumann and continued by Piron. It consists in considering the structure of pure states as determined by the structure of the set of