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On the Borel Structure of C^*-Algebras

(With an Appendix by R. V. KADISON)

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Abstract. We provide a method of embedding a C^*-algebra $\mathscr{A}$ in a C^*-algebra $\mathscr{A}^\sim$ called its σ-envelope. $\mathscr{A}^\sim$ is contained in the enveloping algebra of $\mathscr{A}$ but is generally much smaller, and if $\mathscr{A}$ is commutative with identity then $\mathscr{A}^\sim$ can be identified with the algebra of bounded Baire functions on the spectrum of $\mathscr{A}$. The main result is to completely determine the structure of $\mathscr{A}^\sim$ for all separable G. C. R. algebras $\mathscr{A}$. This provides a good basis for a non-commutative theory of probability.

1. Introduction

We obtain a canonical procedure for embedding a C^*-algebra $\mathscr{A}$ in a C^*-algebra $\mathscr{A}^\sim$ which has the property that every self-adjoint element of $\mathscr{A}^\sim$ has a spectral decomposition in $\mathscr{A}^\sim$. The algebra $\mathscr{A}^\sim$ is a subalgebra of the enveloping algebra $\mathscr{A}^{**}$ and in the case where $\mathscr{A}$ is a commutative C^*-algebra with identity, $\mathscr{A}^\sim$ can be identified with the C^*-algebra of all bounded Baire functions on the spectrum of $\mathscr{A}$. In the general case our work can be regarded as providing a basis for a non-commutative version of measure theory.

We undertake a close analysis of the structure of the algebra $\mathscr{A}^\sim$ and show that it is closely related to the Borel structures of the spectrum $\hat{\mathscr{A}}$ of $\mathscr{A}$. In the case where $\mathscr{A}$ is a separable G.C.R. algebra we can explicitly write down the structure of $\mathscr{A}^\sim$ (Theorem 4.5). This provides us with a non-commutative generalization of the idea of a standard Borel space [9]. As a particular application we analyse the space of finite positive traces on a separable G.C.R. algebra.

If $\mathscr{A}$ is a separable G.C.R. algebra, the set $\mathscr{P}$ of projections in $\mathscr{A}^\sim$ forms a σ-complete orthocomplemented lattice. In a further paper we shall show how this observation allows us to relate our theory to Mackey's formulation of quantum mechanics [10], by letting $\mathscr{P}$ be the partially ordered set of questions in some quantum mechanical system. Slightly different work along these lines is being done by R. J. PLYMEN [12].

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