Commun. math. Phys. 7, 1—20 (1968)

Ergodic States in a Non Commutative Ergodic Theory

S. Doplicher* and D. Kastler**

Institut des Hautes Etudes Scientifiques
91 Bures-sur-Yvette — France

Received July 26, 1967

Abstract. Using the Godement mean $\mathcal{M}$ of positive-type functions over a group G, we study "$\mathcal{M}$-abelian systems" $\{\mathfrak{A}, \alpha\}$ of a C^*-algebra $\mathfrak{A}$ and a homomorphic mapping α of a group G into the homomorphism group of $\mathfrak{A}$. Consideration of the Godement mean of $f(g) U_g$ with f a positive-type function over G and U a unitary representation of G first yields a generalized mean-ergodic theorem. We then define the Godement mean of $f(g)\, \pi(\alpha_g(A))$ with $A \in \mathfrak{A}$ and π a covariant representation of the system $\{\mathfrak{A}, \alpha\}$ for which the G-invariant Hilbert space vectors are cyclic and study its properties, notably in relation with ergodic and weakly mixing states over $\mathfrak{A}$. Finally we investigate the "discrete spectrum" of covariant representations of $\{\mathfrak{A}, \alpha\}$ (i.e. the direct sum of the finite-dimensional subrepresentations of the associated representations of G).

§ 1. Introduction

A number of recent papers [1, 2, 3, 4, 5, 6, 6a] concern themselves with "asymptotically abelian systems" i.e. pairs of a C^*-algebra $\mathfrak{A}$ and a locally compact group G together with a homomorphism $g \to \alpha_g$ of G into the automorphism group of $\mathfrak{A}$ such that one has an "asymptotic abelian property": the commutator

$$\alpha_g(A) \cdot B - B \cdot \alpha_g(A)$$

tends to zero for $g \in G$ tending to infinity for all elements $A, B \in \mathfrak{A}$ (there are different ways of stating this condition corresponding to different choices of topologies — more general conditions can also be stated for general, non topological, groups). The consideration of such asymptotically abelian systems originates in algebraic field theory [10, 11] where the C^*-algebra is that of quasi-local observables (i.e. bounded observables performed within bounded space-time regions together with their norm limits). The group G corresponds to some invariance group of

* On leave of absence from Istituto di Fisica „G. Marconi" Piazzale delle Scienze 5 — Roma.

** Permanent address: Département de Physique Théorique, Faculté des Sciences, Place Victor Hugo — 13 Marseille.