Commun. math. Phys. 6, 248—261 (1967)

Noether Equations and Conservation Laws*

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Received June 18, 1967

Abstract. The purpose of the paper is to present a rigorous derivation of the relation between conservation laws and transformations leaving invariant the action integral. The (space-)time development of a physical system is represented by a cross section of a product bundle M. A Lagrange function is defined as a mapping $L : \overline{M} \to \mathbf{R}$, where $\overline{M}$ is the bundle space of the first jet extension of M. A symmetry transformation is defined as a bundle automorphism of M, carrying solutions of the Euler-Lagrange equation into solutions of the same equation. An important class of symmetry transformations is that of generalized invariant transformations: they are defined by specifying their action on the Euler-Lagrange equation. The generators of generalized invariant transformations are solutions of a system of linear, homogeneous partial differential equations (Noether equations). The set of all solutions of these equations has a natural structure of Lie algebra. In a simple manner, the Noether equations give rise to differential conservation laws.

1. Introduction

The nature of the connection between symmetries and the existence of conserved quantities is an intriguing physical problem. The theory of this connection, as it appears in classical physics, constitutes one of the most beautiful chapters of mathematical physics. The fundamental work on this problem was done by EMMY NOETHER in 1918 [1]. Since then, a rather large number of papers have appeared on this subject. They contain either generalizations of Noether's results [2] or their application to particular physical theories. Little work has been done on a precise statement and proof of the basic theorems relating properties of invariance to conservation laws.

The formulation that these theorems have been given until recently can be summarized as follows. To alleviate the exposition, we make here a number of simplifying assumptions. Let n and N be positive integers and consider a physical system whose history (space-time development)

* Supported by Air Force Office of Scientific Research and Aeronautical Research Laboratories.

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