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The Heisenberg Ferromagnet as a Quantum Field Theory

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Abstract. We consider a lattice of spin $\frac{1}{2}$ ions, described by the discrete form of the current commutation relations $J_i^\alpha J_{(i)}^\alpha = \frac{1}{2}$, $[J_i^\alpha, J_j^\beta] = i\delta_{ij}\,\varepsilon^{\alpha\beta\gamma} J_i^\gamma$ where $\alpha = 1, 2, 3$ and i label the lattice sites. The algebra is realized as the Clifford algebra $\mathfrak{A}$ over a Hilbert space. The equations of motion are specified by a formal Hamiltonian of the Heisenberg form: $\mathrm{H} = \sum_{i,j} f_{ij} \underline{J}_i \cdot \underline{J}_j$, where $f_{ij} \leqq 0$ and only a finite number Q of ions are linked to any given lattice site. We prove that the Hamiltonian is non-negative in a representation of $\mathfrak{A}$, and has a ground state Ω exhibiting ferromagnetism. The time displacement group acts continuously on $\mathfrak{A}$, inducing automorphisms. $\mathfrak{A}$ is asymptotically abelian with respect to the space translations of the lattice.

The model is an example of an algebraic quantum field theory and possesses a broken symmetry, the rotation group 0(3). The consequent Goldstone theorem is proved, namely, there is no energy gap in the spectrum of H.

1. Introduction and Summary

In this paper we apply the ideas of *local quantum theory* [1—4] to the theory of the Heisenberg ferromagnet [5]. The intention is to discuss the axioms of quantum statistical mechanics [6—8] with reference to this particular model.

Denote by Z^d (where Z is the group of integers) the regular arrays of points in d dimensions, $d = 1, 2, 3$. The lattice is invariant under translations by Z^d. The points of Z^d will be called *lattice sites*. At each lattice site is placed an ion with spin $\frac{1}{2}$. That is, the states of a single ion $i \in Z^d$ can be described by the vectors in a two-dimensional Hilbert space $\mathscr{H}_i$ carrying the fundamental spinor representation of SU_2, the covering group of the rotation group in three dimensions (we use the same description of spin whether the ions are arranged in a chain, a plane or a cube). In the model under consideration the motion of the ions, even the lattice vibrations, are ignored. The "observables" describing an ion i comprise the three Pauli matrices $\sigma_i^1, \sigma_i^2, \sigma_i^3$ where $J_i = \frac{1}{2}\sigma_i$ measures

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