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Asymptotically Abelian Systems

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Abstract. We study pairs $\{\mathfrak{A}, \alpha\}$ for which $\mathfrak{A}$ is a C^*-algebra and α is a homomorphism of a locally compact, non-compact group G into the group of *-automorphisms of $\mathfrak{A}$. We examine, especially, those systems $\{\mathfrak{A}, \alpha\}$ which are (weakly) asymptotically abelian with respect to their invariant states (i.e. $\langle \Phi | A \alpha_g(B) - \alpha_g(B) A \rangle \to 0$ as $g \to \infty$ for those states Φ such that $\Phi(\alpha_g(A)) = \Phi(A)$ for all g in G and A in $\mathfrak{A}$). For concrete systems (those with $\mathfrak{A}$ acting on a Hilbert space and $g \to \alpha_g$ implemented by a unitary representation $g \to U_g$ on this space) we prove, among other results, that the operators commuting with $\mathfrak{A}$ and $\{U_g\}$ form a commuting family when there is a vector cyclic under $\mathfrak{A}$ and invariant under $\{U_g\}$. We characterize the extremal invariant states, in this case, in terms of "weak clustering" properties and also in terms of "factor" and "irreducibility" properties of $\{\mathfrak{A}, U_g\}$. Specializing to amenable groups, we describe "operator means" arising from invariant group means; and we study systems which are "asymptotically abelian in mean". Our interest in these structures resides in their appearance in the "infinite system" approach to quantum statistical mechanics.

Introduction

In the general frame of quantum mechanics the physical observables are described as self-adjoint operators on a Hilbert space $\mathscr{H}$ and the bounded observables (corresponding to bounded operators) therefore generate a C^*-algebra acting on $\mathscr{H}$. The algebraic approach to field theory [1, 2] proposes to consider as physical only the *local* observables i.e. those corresponding to measurements performed within finite regions of space during a finite time. These observables are described mathematically as the self-adjoint elements of an incomplete C^*-algebra whose completion $\mathfrak{A}$, called *the quasi-local algebra*, is considered as the main

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