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On the Continuity of Causal Automorphisms of Space-Time

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Abstract. We prove that any causal automorphism of the (curved or not) space-time in any dimension is continuous.

In a recent paper, ZEEMAN [1] has shown that relativistic invariance is implied by causality alone. More precisely, he has proved that the group of automorphisms of the Minkowski space M_4 preserving the partial ordering $x < y$ (we say that $x < y$ iff $(x-y) \cdot (x-y) > 0$ and $x_0 < y_0$) coincides with the group G consisting of the connected Poincaré group plus space reversal and dilatations. Here $x = (x_0, x_1, x_2, x_3)$ is a vector in M_4 and $x \cdot y$ is the Lorentz scalar product of x and y. Linearity or continuity are not assumed a priori.

ZEEMAN's proof leans essentially on the four-dimensional character of M_4, and, in fact (as pointed out by himself) the result fails for the M_2 space. On the other hand, the (pseudo) euclidean character of the whole of M_4 is also essential for his proof, and thus, as was to be expected, the theorem does not apply to the space-time of general relativity (V_4) either. A less strong result, however, may still be shown to hold in the more general cases. In fact, we are going to prove:

Theorem. *Let the* $(n+1)$*-dimensional Minkowski space with the metric* $x \cdot y = \sum\limits_0^n x_\mu y_\nu g_{\mu\nu}$, $g_{\mu\nu} = \mathrm{diag}(+1, -1, \ldots, -1)$ *be denoted by* M_{n+1}, *and let the space* V_{n+1} *be locally homeomorphic to* M_{n+1}. *If the (bijective) automorphism* f *of* V_{n+1} *is causal, then it is continuous.*

We assume the euclidean topology for M_{n+1}. By locally homeomorphic we mean that for every p in V_{n+1} there exist a x in M_{n+1}, two open neighborhoods of p, x, in V_{n+1}, M_{n+1} denoted by Γ, Γ', and a mapping $\varphi : \Gamma \to \Gamma'$ continuous and one-to-one such that φ^{-1} is also continuous. Finally, to define causal mappings we proceed as follows: it is clear that, due to the topological manifold structure of V_{n+1}, we must define causality in terms of local causality. Thus we begin by defining locally causal mappings: f is locally causal if a) for every point p in V_{n+1} and x in M_{n+1} there exists a Γ around p as above and the open

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