Remarks on Boson Commutation Rules

C. ITZYKSON

Service de Physique Théorique
Centre d'Etudes Nucléaires de Saclay
BP Nᵒ 2 — 91 Gif-sur-Yvette

Abstract. A group theoretical derivation is given of Bargmann's representation of the boson commutation rules in an Hilbert space of analytic functions. Several interesting problems arise in the study of the global representation of the canonical group $S_p(2n, R)$. As a by-product we recover Laguerre-polynomials as spherical functions on the nilpotent Weyl group.

I. Introduction

Several years ago, V. BARGMANN [1] has described a representation of the creation and annihilation operators for bosons in a Hilbert space of analytic functions. We want to show some interesting connections between this construction and the theory of group representations. This appears when one attempts to find a representation of the basic commutation rules of quantum mechanics $[p, q] = -i$ using the device of WEYL. Namely, one introduces a nilpotent group the Lie algebra of which is closely related to the preceding commutation relations. It will be shown that this leads quite naturally to the Bargmann space of entire functions. One may then ask for the global representation of the canonical group in this space. Again this was done by BARGMANN though not in great detail. We observe that the representation splits into two irreducible parts (a fact well known in the study of the harmonic oscillator) each of which is double valued. This appears to have been noticed only recently [2] and it is amusing to note the analogy with the fermion case. This is somehow unexpected, since in contrast with the orthogonal real groups the symplectic ones have an infinitely sheeted covering.

The study of the canonical transformations is made much easier by a mapping on a second Hilbert space of analytic functions in a unit disk. It allows to compare the double valued representation to the known one-valued ones of $SL(2, R)$ [3].

One can easily extend these considerations to N degrees of freedom. The analog of the unit disk is found in this case to be the set of N by N complex symmetric matrices S such that the hermitian matrix $I - \bar{S}S$ is positive definite. We shall denote the set of these matrices by $\mathscr{S}_N$.