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On some Groups of Automorphisms of Physical Observables

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Abstract. Given a weakly continuous one-parameter group of automorphisms of a C^*-algebra $\mathfrak{A}$ of operators on a Hilbert space we show that it is implementable by a strongly continuous one-parameter group of unitary operators belonging to the weak closure of $\mathfrak{A}$, provided that a certain condition — akin to the boundedness from below of the spectrum of the generators — is satisfied.

In recent years, some results have been obtained on special one-parameter groups of automorphisms of a set of observables.

Consider, in particular, the case in which the observables are represented in a Hilbert space $\mathscr{H}$ by the weak closure $\mathfrak{A}^-$ of the union of algebras associated with bounded space-time regions. Let the group of translations in the direction a_μ be induced by a strongly continuous one-parameter group of unitary operators on $\mathscr{H}$, with generator T_{a_μ}. In this framework, ARAKI [1] has shown that if T_{a_μ} has spectrum bounded below and if the lowest point in the spectrum corresponds to at least one eigenvector $\xi \in \mathscr{H}$, then T_{a_μ} is affiliated to $\mathfrak{A}^-$ (in the sense that each projection of T_{a_μ} belongs to $\mathfrak{A}^-$).

When the generator T is bounded, the infinitesimal form of a group of automorphisms is a derivation of $\mathfrak{A}^-$ induced by T[1].

Recently, KADISON [2] has proved the following remarkable theorem:

Theorem α**.** Let $\mathfrak{A}$ be a C^*-algebra[2] of operators on a Hilbert space $\mathscr{H}$ and let $\delta : A \to \delta(A)$ be a derivation of $\mathfrak{A}$. Then there exists a bounded operator T in $\mathfrak{A}^-$ (the weak closure of $\mathfrak{A}$) such that $\delta(A) = [A, T]$ for all $A \in \mathfrak{A}$.

An immediate consequence is

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[1] A derivation of an algebra $\mathfrak{A}$ is a linear map δ from $\mathfrak{A}$ to $\mathfrak{A}$ such that $\delta(AB) = \delta(A) \cdot B + A \cdot \delta(B)$ for all $A, B \in \mathfrak{A}$. If $\mathfrak{A}$ is an algebra of operators on a Hilbert space $\mathscr{H}$ and T is a bounded operator, and if $[A, T] \in \mathfrak{A}$ for all $A \in \mathfrak{A}$, then the map δ_T defined by $\delta_T(A) \equiv [A, T]$ is a derivation of $\mathfrak{A}$, which is said to be induced by T.

[2] For definitions and some results on C^*-algebras, see e.g., DIXMIER, les C^*-algebres et leur representations, Gauthier-Villars, Paris, 1964.