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Canonical Quantization

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Abstract. The dynamical variables of a classical system form a Lie algebra $\mathfrak{G}$, where the Lie multiplication is given by the Poisson bracket. Following the ideas of SOURIAU and SEGAL, but with some modifications, we show that it is possible to realize $\mathfrak{G}$ as a concrete algebra of smooth transformations of the functionals Φ on the manifold $\mathfrak{M}$ of smooth solutions to the classical equations of motion. It is even possible to do this in such a way that the action of a chosen dynamical variable, say the Hamiltonian, is given by the classical motion on the manifold, so that the quantum and classical motions coincide. In this realization, constant functionals are realized by multiples of the identity operator. For a finite number of degrees of freedom, n, the space of functionals can be made into a Hilbert space $\mathscr{H}$ using the invariant Liouville volume element; the dynamical variables F become operators $\hat{F}$ in this space. We prove that for any hamiltonian H quadratic in the canonical variables $q_1 \ldots q_n, p_1 \ldots p_n$ there exists a subspace $\mathscr{H}_1 \subset \mathscr{H}$ which is invariant under the action of $\hat{p}_j$, $\hat{q}_k$ and $\hat{H}$, and such that the restriction of $\hat{p}_j$, $\hat{q}_k$ to $\mathscr{H}_1$ form an irreducible set of operators. Therefore, SOURIAU's quantization rule agrees with the usual one for quadratic hamiltonians. In fact, it gives the Bargmann-Segal holomorphic function realization. For non-linear problems in general, however, the operators $\hat{p}_j$, $\hat{q}_k$ form a reducible set on any subspace of $\mathscr{H}$ invariant under the action of the Hamiltonian. In particular this happens for $H = \frac{1}{2} p^2 + \lambda q^4$. Therefore, SOURIAU's rule cannot agree with the usual quantization procedure for general non-linear systems.

The method can be applied to the quantization of a non-linear wave equation and differs from the usual attempts in that (1) at any fixed time the field and its conjugate momentum may form a reducible set (2) the theory is less singular than usual.

For a particular wave equation $(\square + m^2)\, \phi(x) = \lambda\, \phi^3(x)$, we show heuristically that the interacting field may be defined as a first order differential operator acting on c^∞-functions on the manifold of solutions. In order to make this space into a Hilbert space, one must define a suitable method of functional integration on the manifold; this problem is discussed, without, however, arriving at a satisfactory conclusion.

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