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The Gell-Mann Formula for Representations of Semisimple Groups*

R. HERMANN

Argonne National Laboratory and Northwestern University

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Abstract. A method for constructing representations of non-compact semisimple groups from representations of semidirect product groups is presented. Necessary and sufficient algebraic conditions for the method to work are given, and these are applied to cases of possible interest for the classification of elementary particles.

1. Introduction

Let G be a non-compact, semisimple Lie group with finite center, with K its maximal compact subgroup. Let $\mathbf{G}$ be its Lie algebra, $\mathbf{G} = \mathbf{K} \oplus \mathbf{P}$ its Cartan decomposition, i.e. $[\mathbf{K}, \mathbf{P}] \subset \mathbf{P}$, $[\mathbf{P}, \mathbf{P}] \subset \mathbf{K}$. Let $\mathbf{G}'$ be its contracted Lie algebra: $\mathbf{G}' = \mathbf{K} \oplus \mathbf{P}'$, isomorphic as a vector space with $\mathbf{G}$, and with the same commutation relations, except that

$$[\mathbf{P}', \mathbf{P}'] = 0 \,.$$

In previous work, we have pointed out that many unitary representations of G can be constructed from unitary representions of G' in a simple way. This is important from a practical point of view, since many aspects of the theory of representations are much simpler for groups, such as G', which are semidirect products of compact and Abelian groups. For example, the problem of decomposing their tensor products is comparatively easy (e.g., see [3], Chapter 12).

The Gell-Mann formula is a particulary simple method of expressing the operators of $\mathbf{G}$ in terms of those of $\mathbf{G}'$. In [3], Chapter 18, we gave a set of sufficient conditions that the formula hold, based on the geometric properties of the representations of G'. In this paper we analyze the formula in a more algebraic manner, using Wigner's method [7] for describing the representations of G'. The results to be obtained below are more complete: again we find that the symmetric spaces K/L on which G also acts transitively play the key role.

2. Representations of a group in terms of its contracted group

Let $\mathbf{G}$, $\mathbf{K}$, $\mathbf{P}$ and $\mathbf{P}'$ be as described above. Let Z_u, $1 \leqq u, v, \ldots \leqq m$, and X_i, $1 \leqq i, j, \ldots \leqq n$, be bases of $\mathbf{K}$ and $\mathbf{P}$, respectively, that are

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