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Upper and Lower Limits for the Number of Bound States in a Given Central Potential

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Abstract. Upper and lower limits for the number of bound states in a given central potential are obtained. They imply that for strongly attractive potentials the number of bound states of given angular momentum increases as the square root of the strength of the potential.

1. Introduction

Let $V(r)$ be a central potential such that the integral

$$I = \int_0^\infty dr\, r |V(r)| \tag{1.1}$$

is finite. Note that this integral is dimensionless in the units chosen, $\hbar = 2m = 1$. JOST and PAIS [1] have shown that a necessary condition for the existence of bound states is $I \geqq 1$. Subsequently BARGMANN [2], and later SCHWINGER [3], have derived the more general inequality

$$n_l \leqq I/(2l + 1)\,, \tag{1.2}$$

where n_l is the number of bound states with angular momentum l. They also show that the estimate eq. (1.2) is "best possible in the sense that for a given l potentials may be constructed which have a prescribed number n_l of bound states for that angular momentum and for which I approaches $(2l + 1)\, n_l$ arbitrarily closely" [2]. This of course does not imply that other upper limits for the number of bound states could not be found which, for many potentials, would yield restrictions more stringent than those obtained from Bargmann's inequality eq. (1.2). In fact it should be noted that the potentials which saturate Bargmann's inequality have a rather peculiar shape; for a given n_l, they