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Linking Numbers of Modular Knots

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To: Peter Lax with admiration

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§1. In this note we examine some questions about closed geodesics on the modular surface X which are suggested by the spectacular pictures from Ghys' talk [G] as well as his paper with Leys [G-L]. The discussion below follows closely my letter to Mozzochi [Sa1] and as is done there we will only outline the proofs of the main results. Detailed proofs of the delicate estimates that are needed will appear in Mozzochi's article [Mo].

First we review the results from [G]. He shows that the non-compact 3-dimensional homogeneous quotient space $Y = SL_2(\mathbb{R})/SL_2(\mathbb{Z})$ is homeomorphic to the 3-sphere S^3 with the trefoil knot τ removed. Y carries a number of flows and corresponding non-vanishing vector fields and in particular the diagonal flow $\mathcal{G}_t$, for $t \in \mathbb{R}$,

$$\mathcal{G}_t(y\, SL_2(\mathbb{Z})) = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} y\, SL_2(\mathbb{Z})\,. \tag{1}$$

This flow corresponds to the geodesic flow on the modular surface $X = \mathbb{H}/\Gamma$ with $\Gamma = PSL_2(\mathbb{Z})$ and the primitive (i.e. once around) closed orbits of $\mathcal{G}_t$ correspond to oriented primitive (or "prime") closed geodesics on X. Such a periodic orbit of $\mathcal{G}_t$ yields a knot in $S^3 - \tau$. It is known that these primitive closed orbits correspond to primitive hyperbolic conjugacy classes $\{A\}_\Gamma$ of elements A in Γ (see [He]). A is primitive if it is not a nontrivial power of an element B in Γ and A is hyperbolic means that $|\text{trace}\,(A)| = t(A) > 2$. In this way to each such $\{A\}_\Gamma$ we get a knot k_A in $S^3 - \tau$.

Figure 1 is taken from [G-L]. It depicts six knots corresponding to the A's indicated and how these wind around the trefoil. The question raised in [G] is to understand the function which takes $\{A\}_\Gamma$ to k_A. For example, which knots k_A arise in this way and how do the linking numbers of k_A with τ vary with A? Ghys establishes some very interesting things about these knots. Firstly, that the set of such knots, dubbed "modular knots", coincides with the set of "Lorenz knots". The latter are the knots which are primitive periodic orbits of the non-linear