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NUMERICAL REPRESENTATIONS OF A UNIVERSAL SUBSPACE FLOW FOR LINEAR PROGRAMS*

P.-A. ABSIL†

Abstract. In 1991, Sonnevend, Stoer, and Zhao [Math. Programming 52 (1991) 527–553] have shown that the central paths of strictly feasible instances of linear programs generate curves on the Grassmannian that satisfy a universal ordinary differential equation. Instead of viewing the Grassmannian $\mathrm{Gr}(m,n)$ as the set of all $n\times n$ projection matrices of rank m, we view it as the set $\mathbb{R}_*^{n\times m}$ of all full column rank $n\times m$ matrices, quotiented by the right action of the general linear group $\mathrm{GL}(m)$. We propose a class of flows in $\mathbb{R}_*^{n\times m}$ that project to the flow on the Grassmannian. This approach requires much less storage space when $n \gg m$ (i.e., there are many more constraints than variables in the dual formulation). One of the flows in $\mathbb{R}_*^{n\times m}$, that leaves invariant the set of orthonormal matrices, turns out to be a particular version of a matrix differential equation known as Oja's flow. We also point out that the flow in the set of projection matrices admits a double bracket expression.

Keywords: Linear programming, Grassmannian, Grassmann manifold, Stiefel manifold, ordinary differential equation, Oja's flow, double bracket flow.

Mathematics Subject Classification: 37N40 Dynamical systems in optimization and economics; 90C05 Linear programming

1. Introduction. One of Roger Brockett's major contributions to date has been to propose and analyze the matrix differential equation [Bro91]

$$H'(t) = [H(t),[H(t),N]], \quad H(0) = H_0, \tag{1}$$

where N and H_0 are $n\times n$ real symmetric matrices and $[A,B] := AB - BA$ denotes the matrix commutator. This matrix flow belongs to a class of flows on manifolds that realize computational algorithms. It is able to solve the eigenvalue problem of a symmetric matrix A (see [Bro91], or [HM94, §2.1], [Deh95, §4.2]): to this end, choose $H_0 = A$ and $N = \mathrm{diag}(\mu_1,\dots,\mu_n)$ with $\mu_1 > \dots > \mu_n$; then $\lim_{t\to\infty H(t)} H(t)$ exists and is a diagonal matrix, whose diagonal elements are the eigenvalues of A since the flow is isospectral (i.e., the spectrum of H does not vary along the trajectory). The differential equation (1) is also capable of sorting lists: if N is chosen, for example, as $\mathrm{diag}(1,2,\dots,n)$, then for almost all orthogonal $n\times n$ matrices Θ and for $H(0) = \Theta^T[\mathrm{diag}(\lambda_1,\lambda_2,\dots,\lambda_n)]\Theta$, the solution of (1) will approach $H(\infty) = \mathrm{diag}(\lambda_{\pi(1)},\lambda_{\pi(2)},\dots,\lambda_{\pi(n)})$ with the final list sorted by size. Brockett also

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†Département d'ingénierie mathématique, Université catholique de Louvain, B-1348 Louvain-la-Neuve, Belgium (www.inma.ucl.ac.be/∼absil).