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ON BROCKETT'S NECESSARY CONDITION FOR STABILIZABILITY AND THE TOPOLOGY OF LIAPUNOV FUNCTIONS ON $\mathbb{R}^N$*

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Abstract. In [2], Roger Brockett derived a necessary condition for the existence of a feedback control law asymptotically stabilizing an equilibrium for a given nonlinear control system. The intuitive appeal and the ease with which it can be applied have made this criterion one of the standard tools in the study of the feedback stabilizability of nonlinear control systems. Brockett's original proof used an impressive combination of Liapunov theory and algebraic topology, in part to cope with a lacuna in our understanding of the topology of the sublevel sets of Liapunov functions. In [33], F. W. Wilson, Jr. extended the converse theorems of Liapunov theory to compact attractors and proved some fundamental results about the topology of their domain of attraction and the level sets of their Liapunov functions. In particular, Wilson showed that the level sets $M_c = V^{-1}(c)$ are diffeomorphic to S^{n-1} for $n \neq 4, 5$ using the proof of the generalized Poincaré Conjecture of Smale. He observed that the excluded cases would from the validity of the Poincaré Conjecture in dimension 3 and 4 and showed that, for $n = 5$, the assertion $\partial M_c \simeq S^4$ would imply the Poincaré Conjecture for 4-manifolds. Of course, the topological Poincaré Conjecture for S^4 was subsequently proved by Freedman in 1980 and with the remarkable recent solution by Perelman of the classical Poincaré Conjecture, Wilson's Theorem now holds for all n.

In this paper we describe the *sublevel*, and therefore as a corollary the level, sets of proper smooth functions $V : \mathbb{R}^n \to \mathbb{R}$ having a compact set $\mathcal{C}(V)$ of critical points. Among the main results in this paper is the assertion that an arbitrary sublevel set $\mathcal{M}_c = V^{-1}[0, c]$ of such a function is homeomorphic to $\mathbb{D}^n$, the unit disk. For $n = 2$, this assertion is a consequence of the Schönflies Theorem, a classical enhancement of the Jordan Curve Theorem. For arbitrary n it follows from the generalized Schönflies Theorem of Mazur and Brown, from [33] and from the verification of the Poincaré Conjecture in all dimensions by Perelman, Freedman and Smale. We also describe the smooth structure of $\mathcal{M}_c$ and its boundary, generalizing the results of [33].

This result has several corollaries. In particular, using the Brouwer Fixed Point Theorem this gives a straightforward proof of Brockett's criterion and some of its enhancements to global attractors. These results in turn imply a new necessary condition for Input-to State Stability with respect to a compact set and an extension of Brockett's Theorem to the practical stabilizability of equilibria. Our main results can be further enhanced using the Poincaré-Hopf Theorem and, in this way, also lead to a streamlined version of Coron's proof [7] that Brockett's Theorem holds for continuous feedback laws, using a classical topological argument on the unit disc $\mathbb{D}^n$.

1. Introduction.

Consider a control system

$$\dot{x} = f(x, u) \quad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m \tag{1.1}$$

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