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Foundations of analysis over surreal number fields, by N. L. Alling. North-Holland Mathematics Studies, vol. 141, North-Holland, Amsterdam, 1987, xvi + 374 pp., $66.75. ISBN 0-444-70226-1

The field **R** of real numbers may be characterized abstractly as a Dedekind-complete ordered field. Moreover, **R** is real-closed and by Tarski's theorem it shares its first-order properties with all other real-closed fields, so to distinguish it from such fields one has to invoke higher-order properties (like Dedekind-completeness). The author has set himself the task of studying these fields. The usual methods involve assumptions equivalent to a form of the generalized continuum hypothesis; here he follows the route taken by Conway in his account "On numbers and games" (London Math. Soc. Monographs No. 6, Academic Press, 1976) to construct such fields and to describe the beginnings of analysis in the new setting.

Conway's method of building up number systems may be regarded as Dedekind cuts taken to extremes. His basic principle states that "If L, R are two sets of numbers and no member of L is $\geq$ any member of R, then $\{L|R\}$ is a number. All numbers are constructed in this way." Starting from $0 = \{|\}$ one obtains in a countable number of steps all dyadic numbers $m2^{-n}$; by continuing transfinitely one reaches all real numbers and indeed can go beyond **R** to construct a Field **No**, i.e. a proper class whose elements admit the operations and satisfy the laws of fields. The members of **No** are the *surreal numbers*. The pairs $\{L|R\}$ used in the construction process are called *Conway cuts* in the ordered set X that is being considered. The special case where L, R form a partition of X into nonempty sets is the familiar Dedekind cut. An intermediate notion, where L or R may be empty (but their union is still X) was introduced in 1954 by N. Cuesta Dutari. Such Cuesta Dutari cuts are used here to construct η_ξ-fields, i.e. real-closed fields that are η_ξ-sets. For $\xi = 0$ such fields are of course well known, e.g. **R**, or the subfield of all real algebraic numbers, but η_ξ-fields for $\xi > 0$ are harder to construct. They are fields displaying a high degree of density and any ordered field whose cardinal is bounded by ω_ξ can be embedded in an η_ξ-field; thus every such field can be embedded in the 'universal' η_ξ-field ξ**No**.

The author's aim may be described as a study of analysis over ξ**No**. For a satisfactory development he has to work with a ξ-topology. This is a modification of the usual definition in that the family of all open sets is closed under finite intersections and unions of fewer than ω_ξ sets (ω_ξ being regular). Notions such as ξ-continuous, ξ-compact, ξ-connected etc. are introduced and it is reassuring to find that addition and multiplication are ξ-continuous and the ξ-connected subsets are just the intervals. In this terminology an η_ξ-set is just an ordered set which is ξ-complete. In fact much of this development is carried out generally for η_ξ-sets and later specialized to apply to surreal fields.