BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 17, Number 2, October 1987

EXOTIC KNOTTINGS OF SURFACES IN THE 4-SPHERE

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1. The main result.

THEOREM. *There exists an infinite series* $S_1, S_2, \ldots$ *of smooth submanifolds of* S^4 *such that*:

(1) *for any* i, j *the pairs* (S^4, S_i), (S^4, S_j) *are homeomorphic,*

(2) *for any* $i \neq j$ *the pairs* (S^4, S_i), (S^4, S_j) *are not diffeomorphic,*

(3) *each* S_n *is homeomorphic to the connected sum* $\#_{10}\mathbf{R}P^2$ *of* 10 *copies of the projective plane,*

(4) $\pi_1(S^4 \backslash S_n) = \mathbf{Z}_2$,

(5) *the normal Euler number* (*with local coefficients*) *of* S_n *in* S^4 *is* 16.

Actually we show instead of (1) a slightly stronger result, namely, that there are smooth isomorphisms φ_i of tubular neighborhoods of S_i and S_1 which can be extended to homeomorphisms of the exterior. But according to (2) $\varphi_j^{-1} \circ \varphi_i$ cannot be extended to a diffeomorphism of the exteriors. This is surprising, as there is no analogous result in other dimensions. Let N be a closed smooth submanifold of a closed manifold M of dimension $\neq 4$. Let U be a smooth tubular neighborhood. Then there are only finitely many diffeomorphism types rel. boundary of smooth manifolds X with $\partial X = \partial U$ and X homeomorphic to $M - \mathring{U}$. If $\dim M = 3$ the number of diffeomorphism types is 1 and if $\dim M \geq 5$ the number of smoothings rel. boundary (which is an upper bound for the number of diffeomorphism types) is finite by [**KS**].

In fact we describe an infinite family $F_1, F_2, \ldots$ of smooth submanifolds of S^4 satisfying conditions (2)–(5) of the Theorem, and we prove that there are only finitely many homeomorphism types of (S^4, F_n) in the sense described above.

The F_n's are obtained from a fixed smooth submanifold $F \subset S^4$ by a family of new knotting constructions. F is the obvious simplest submanifold satisfying the conditions (3), (4), and (5): the pair (S^4, F) is the connected sum of the standard pair $(S^4, \mathbf{R}P^2)$ (with normal Euler number -2) and nine copies of it with the orientation of S^4 reversed.

Our knotting constructions can be applied to "smaller" submanifolds, e.g. the Klein bottle with normal Euler number 0 and the torus, which are standardly embedded in S^4. The only thing we fail to prove in these situations is the nonexistence of diffeomorphisms.

Received by the editors June 15, 1987.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 57Q45, 57R55, 57N35; Secondary 57M25, 57M12.

Second author partially supported by the DFG and the Academy of Science of the USSR.