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CONTINUED FRACTALS AND THE SEIFERT CONJECTURE

BY JENNY HARRISON[1]

In 1950 Herbert Seifert posed a question today known as the Seifert Conjecture:

"Every C^r vector field on the three-sphere has either a zero or a closed integral curve."

Paul Schweitzer published his celebrated C^1 counterexample in 1971 [**Sch**]. We show how to obtain a $C^{3-\varepsilon}$ counterexample X by using techniques from number theory, analysis, fractal geometry, and differential topology [**H1** and **H2**]. X is C^2 and its second derivative satisfies a $(1-\varepsilon)$-Hölder condition.

1. Continued fractions and quasi-circles. Any irrational number α, $0 < \alpha < 1$, can be expressed as a continued fraction

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

where the a_i are positive integers. One writes $\alpha = [a_1, a_2, a_3, \ldots]$. The truncation $[a_1, \ldots, a_n] = p_n/q_n$ is the best approximation to α among all rational numbers p/q with $0 < q \leq q_n$. The growth rate of the a_i tells "how irrational" $\alpha = [a_i]$ is. At one extreme is the Golden Mean, $\gamma = [1, 1, \ldots]$); at the other are Liouville numbers such as $\lambda = [1^{1!}, 2^{2!}, 3^{3!}, \ldots]$. The former is "very irrational" while the latter is "almost rational".

To study α dynamically it is standard to consider R_α, the rigid rotation of the circle S^1 of unit length through angle α. Choose $x \in S^1$ and consider its R_α-orbit $O_\alpha(x)$. Since α is irrational, $O_\alpha(x)$ is dense in S^1. But how is it dense? For Liouville λ, $O_\lambda(x)$ contains long strings $\{R_\lambda^n(x), R_\lambda^{n+1}(x), \ldots, R_\lambda^m(x)\}$ that are poorly distributed. They "bunch up". In contrast, the Golden Mean's orbit distributes itself fairly evenly throughout S^1.

Unfortunately, it is hard to distinguish visually (and hence geometrically) between bunched-up dense orbits and well distributed ones. After many iterates, the orbit picture becomes blurred. This is due in fact to the picture's being drawn on the circle. As a remedy, we "unfold" S^1 onto a canonically constructed curve Q_α in the 2-sphere S^2 as follows.

Choose a "Denjoy" projection $\rho\colon S^1 \to S^1$; that is, ρ is onto and continuous, $\rho^{-1}\langle n\alpha\rangle$ is an interval I_n for all $n \in \mathbf{Z}$, the I_n are disjoint, and ρ is 1-1

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