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# FIXED POINTS OF SURFACE HOMEOMORPHISMS

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For a self-map $f$ of a compact connected polyhedron $X$, the Nielsen number $N(f)$ is defined to be the number of essential fixed point classes (see [1] for an introduction to Nielsen fixed point theory). It is a classical theorem of Wecken [7] that $N(f)$ is a lower bound of the number of fixed points for all maps homotopic to $f$, and that if $X$ is a manifold of dimension $\geqslant 3$ this lower bound is always realizable (see also [1]). It is now known [3] that $N(f)$ is realizable if $X$ has no local cut points and $X$ is *not* a surface. The realizability problem for maps on surfaces is still open.

Historically, the theory of fixed point classes originated with Nielsen's study [5] of surface homeomorphisms. The purpose of this work is to examine the realizability of the Nielsen number of a surface homeomorphism as the least number of fixed points in an isotopy class. The same problem for higher-dimensional manifolds is open.

MAIN THEOREM. *Let $M^2$ be a compact surface, closed or with boundary. Let $\varphi$: $M^2 \longrightarrow M^2$ be a homeomorphism. Then $\varphi$ is isotopic (through embeddings) to an embedding which has $N(\varphi)$ fixed points. If, in addition, no boundary component of $M^2$ is mapped onto itself by $\varphi$ in an orientation-reversing manner, then $\varphi$ is isotopic (through homeomorphisms) to a homeomorphism having $N(\varphi)$ fixed points.*

The reason why we have to be contented with embeddings instead of homeomorphisms in the presence of an orientation-reversed invariant boundary component has already been explained in [4].

Our proof is based on Thurston's theory of surfaces [6, 2] which enables us to visualize a representative from each isotopy class.

For surfaces with $\chi(M^2) \geqslant 0$, the truth of the Theorem can be checked case by case. In the general case $\chi(M^2) < 0$, Thurston [6] (see also [2]) tells us that every diffeomorphism $\varphi$ is isotopic to a "diffeomorphism" $\varphi'$ which is either (1) an isometry with respect to a hyperbolic metric, or (2) a pseudo-Anosov "diffeomorphism", or (3) reducible in the sense that $M^2$ can be cut into simpler parts along an invariant set of disjoint simple closed curves, such that on each part, $\varphi'$ is of type (1) or (2) in the complement of a collar around its boundary.

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